



W. E. C. C. C.

INTRODUCTORY ALGEBRA

A FIRST COURSE
FOR SECONDARY SCHOOLS
AND JUNIOR HIGH SCHOOLS

BY

ALAN JOHNSON

HEAD OF MATHEMATICS DEPARTMENT
BARRINGER HIGH SCHOOL, NEWARK, N. J.

AND

ARTHUR W. BELCHER

HEAD OF MATHEMATICS DEPARTMENT
EAST SIDE COMMERCIAL AND MANUAL TRAINING HIGH SCHOOL
NEWARK, N. J.

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PREFACE

This book is written for boys and girls who are beginning the study of Algebra. It has been growing for several years out of an increasing realization of the needs of those who have been in our class rooms. It aims to embody the new spirit in mathematics as set forth in the report of the National Committee on Mathematical Requirements and as defined by the College Entrance Examination Board. The authors feel that they have gone as far in the direction of general mathematics as may advantageously be done, if we are not to lose the value of definitely ordered and progressive work in Algebra.

Some features of the book are:

1. The large number of examples and questions, about 4000 in all, which enable the teacher to develop each new topic or principle with an abundance of carefully graded material. It is much easier to make omissions than it is to plan and dictate whatever might be necessary in overcoming the deficiencies of an inadequate test.

2. The gradual and thorough development of the conception of a formula: its usefulness in abbreviating simple rules; exercises; in symbolic language; the substitution of numerical values; the translation of formulas into rules; the making of formulas based upon

each of the four fundamental relations in turn; changing the subject of a formula to any letter; etc.

3. The use of four-place mathematical tables. So important has become the use of tables by mechanics, engineers and business men that ability to use them forms one of the legitimate and desirable goals in modern algebra.

4. A consistent use of checks throughout the book, with a more logical and effective arrangement of the check than has heretofore been the case.

5. Frequent arithmetical questions and examples. The arithmetic helps to make clear the related processes of algebra, and the algebra in turn illuminates the arithmetic. It is the intention to keep alive a friendly feeling for arithmetic and not altogether to lose sight of simple fractions and decimals, per cents and interest.

6. The many graphs, associated with the particular subject matter which they illustrate and amplify;—negative numbers, tabulations and formulas, ratio, angles, simultaneous equations, etc.

7. The informal work in geometry which is utilized to illustrate and apply various topics in algebra, and to furnish a basis for the trigonometry.

8. The simple presentation of the elementary facts of trigonometry. This chapter has been found to be not more difficult than quadratic equations, for instance, but of greater worth. The chapter on quadratic equations may be used at the option of the teacher.

9. The review exercises and specimen tests, at the

places where experience has shown that they are needed.

We wish to acknowledge our indebtedness to our fellow workers in the Newark high schools, who have studied and tried out the various chapters, and especially to David E. Amidon who has read all the manuscript and given many helpful suggestions.

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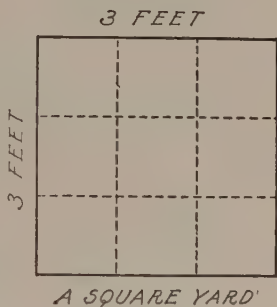
INTRODUCTORY ALGEBRA

CHAPTER I

THE USE OF SIMPLE FORMULAS

SQUARES

The large square in the diagram, representing a square yard, is divided into smaller squares, each representing a square foot. How many of the small squares are there? How many square feet are there in one square yard? What is the area, in square inches, of a square three inches on a side?



EXERCISE 1

1. The side of a square is 8 inches in length; what is its area in square inches?

2. How many square inches are there in the area of a square whose side measures 12 inches? How many square inches are there in one square foot?

3. Make a complete statement telling the method or rule which you use in finding the area of a square when you know the length of its side.

4. Using the rule which you gave in Example 3, find the area of a square if the length of its side is (a) 20 feet; (b) $\frac{1}{2}$ inch.

5. A square is drawn, one inch on a side. If in one corner of it you draw a square $\frac{1}{2}$ inch on a side, what part of the square inch is it?



6. The square centimeter in the diagram measures 10 millimeters on a side. Each of the little squares is 1 square millimeter. How many square millimeters are there in the square centimeter?

The Square of a Number

Because the product obtained by multiplying the length of the side of a square by itself gives the area of the square, *the product obtained by multiplying any number by itself is called the square of the number*. For instance, since 9 is obtained by multiplying 3 by 3, 9 is called the square of 3; it is written 3^2 (we read it "three square") and means 3 times 3. The square of 12 is written 12^2 , meaning 12×12 .

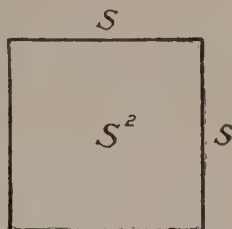
EXERCISE 2

1. What is the value of 12^2 ?
2. Copy and fill in the blanks: "The square of 7 is written — and is equal to 7 times 7, or —."
3. Make a similar statement for the square of 6.
4. Copy and fill in the blanks in the following:
 - (a) $25^2 = \text{--- times ---, or ---.}$
 - (b) $(\frac{3}{4})^2 = \text{---} \times \text{---} = \text{---.}$
 - (c) $1.5^2 = \text{---} \times \text{---} = \text{---.}$
5. What is the square of 8? the square of 40?
6. Find the value of 5^2 , of 6^2 , and of $5^2 + 6^2$.
7. Find the value of $(\frac{11}{2})^2$ and of 5.5^2 .
8. A rod is $5\frac{1}{2}$ yards long. How many square yards are there in a square rod?
9. Copy and fill in the blank spaces:

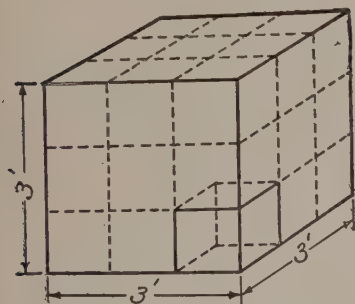
— sq. inches	= 1 sq. ft.
— sq. feet	= 1 sq. yd.
— sq. yards	= 1 sq. rod

In algebra we often represent a quantity, such as the side of a square, by a single letter. In this diagram we use the letter s to represent the number of inches in the length of a side of the square. If s is 10 the area of the square is $s \times s$, which is 10×10 or 100 square inches. The product $s \times s$ means the square of s and is written s^2 (we read it "s square").

10. Copy and fill in the blanks: "If $s = 13$, $s^2 = 13^2$ or — times —; therefore $s^2 =$ —."



11. Complete: If $s = 30$, $s^2 =$ — $\times$ — = —.
12. What is the value of s^2 if $s = 16$?
13. Find the value of s^2 if $s = 1.1$.
14. If a has the value 9 what is the value of a^2 ?
15. Find the value of $a^2 + b^2$ if $a = 3$ and $b = 2$.
16. What is the value of $a^2 + b^2 + c^2$ if $a = 4$, $b = 3$, and $c = 1$?
17. Find the value of $R^2 - a^2$ when $R = 5$ and $a = 2$.
18. Show that $c^2 = a^2 + b^2$ when $a = 12$, $b = 5$, and $c = 13$.
19. If you know that $y = m^2 + n^2$, find the value of y when $m = 1.2$ and $n = 1.6$.



A CUBIC YARD

CUBES

Each edge of this cube is 3' long (3' is a short way of writing 3 feet). How many cubic feet does the cube contain? How many cubic feet are there in a cubic yard? How many cubic inches are there in a cubic foot?

$$(12 \times 12 \times 12 = ?)$$

EXERCISE 3

1. If the edge of a cube is 2'' in length what is its volume in cubic inches? (2'' is a short way of writing 2 inches.)

2. State the rule or method you use in finding the volume of a cube.

3. Use the rule you have just stated to find the volume of a cube each of whose edges is $\frac{1}{2}$ ''.

The Cube of a Number

Because the product of $12 \times 12 \times 12$ gives the volume of a cube whose edge is 12, this product is called the cube of 12 and is written 12^3 (we read it "12 cube"). For instance, since 27 is obtained by multiplying $3 \times 3 \times 3$, 27 is called the cube of 3 and is written 3^3 .

4. What is the value of 12^3 ?

5. Copy the following statement, filling in the blanks: "The cube of 4 is written — and is equal to — $\times$ — $\times$ — or —."

6. Make a similar statement for the cube of 10.

7. Fill in: $5^3 = \text{---} \times \text{---} \times \text{---} = \text{---}$.

8. Find the cube of $\frac{1}{3}$, working it out by the same plan as Example 7.

9. A cubic foot is what part of a cubic yard?

Just as s^2 is a shorthand way of writing $s \times s$, so s^3 is a shorthand way of writing $s \times s \times s$. If $s = 6$ then

$$\begin{aligned}s^3 &= s \times s \times s \\ &= 6 \times 6 \times 6 \\ &= 216\end{aligned}$$

10. Write in a shorter way (a) $m \times m$; (b) $a \times a \times a$.

11. Complete: "If $s = 9$, $s^3 = \text{---} \times \text{---} \times \text{---} = \text{---}$."

12. Find the value of s^3 (a) if s is equal to 8

(b) if $s = \frac{1}{4}$

(c) if $s = .5$

13. What is the value of s^3 if $s = 1$?

14. How many cubic feet of air are there in a room 11' high, 11' long, and 11' wide?

15. What is the value of $a^3 - b^3$ if $a = 4$, $b = 3$?

16. Find the value of $m^3 + n^3$ if $m = 8$, $n = 2$.

17. If $y = a^3 + b^3 + c^3$, find the value of y if $a = 1.1$, $b = 1.2$, and $c = 1.3$.

18. Draw three columns as below. Write the numbers from 1 to 12 in the first column and opposite each number write its square in the second column and its cube in the third column.

NUMBER n	SQUARE n^2	CUBE n^3
1		
2		
3		
4		
etc.		

POWERS

Such numbers as 5^3 , s^2 , or 3^4 are called *powers* and the small number which indicates the power is called an *exponent*. Thus 5^3 means the third power of 5, and the exponent 3 says that three 5's must be multiplied together, giving $5 \times 5 \times 5$ or 125. The fourth power of 3 would be written 3^4 and means that four 3's are to be multiplied together, giving $3 \times 3 \times 3 \times 3$ or 81. In the same way a^5 (which we read " a to the fifth power") would mean $a \times a \times a \times a \times a$.

EXERCISE 4

1. Copy and complete this sentence: "The cube of 4 is written —, and is the same as the — power of 4; it is equal to — $\times$ — $\times$ — or —."

2. Make a similar statement for the square of 9.

3. 2^4 ("2 to the fourth power") = $2 \times 2 \times 2 \times 2 = ?$

4. $5^4 = \text{—} \times \text{—} \times \text{—} \times \text{—} = ?$

5. Find the value of 2^5 .

6. With the help of exponents write each of the following in a shorter way:

(a) $3 \times 3 \times 3$

(d) $a \times a$

(b) 5×5

(e) $s \times s \times s$

(c) $10 \times 10 \times 10 \times 10$

(f) $m \times m \times m \times m$

7. Find the value of 4^4 .

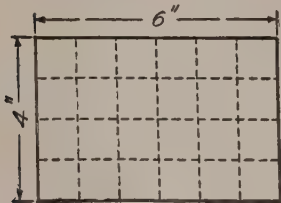
8. What is the value of n^5 if $n = 3$?

9. If $n = 5$ find the value of n^2 ; n^3 ; n^4 .

10. Find the values of n^2 , n^3 , and n^4 if $n = 1$.

RECTANGLES

To find the number of square inches in the area of a rectangle we use the method well known in arithmetic,



diagram?

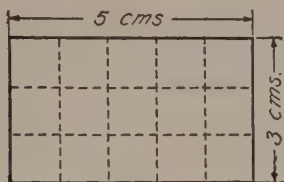
which is to multiply the number of inches in the length by the number of inches in the width. The rule is usually shortened and stated: *The area of a rectangle equals the length times the width.* What is the area of the rectangle in the

In algebra we abbreviate the rule still further, representing the area of the rectangle by the letter A , the word "equals"

by $=$; and the length and width by their initial letters l and w . The rule becomes

$$A = l \times w$$

This abbreviated form of the rule is called a formula.



Example: What is the area of a rectangle 5 centimeters long and 3 centimeters wide?

Solution:

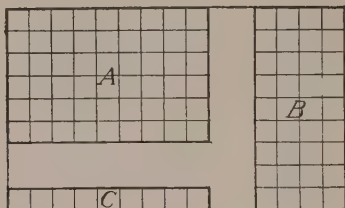
$$\begin{array}{ll} l = 5 & A = l \times w \\ w = 3 & = 5 \times 3 \\ A = ? & = 15 \text{ square centimeters} \quad \text{Ans.} \end{array}$$

EXERCISE 5

1. What is the area in square inches of a rectangle which is 15 inches in length and 10 inches in width?
2. How many square rods are there in a rectangular field 20 rods long and 8 rods wide? (An acre.)
3. How many square yards are there in the ceiling of a schoolroom 10 yards long and 8 yards wide?
4. How many square centimeters are there in a rectangular card 18 centimeters long and 11 centimeters wide?
5. Using the formula $A = l \times w$ find A if $l = 8\frac{1}{2}''$ and $w = 4''$.
6. Find the area of a township which is 6 miles square ($l = 6$, $w = 6$, $A = ?$).

7. If $l = 3.2''$ and $w = 1.5''$ what is A ?

8. Into how many small squares, each $1''$ square, could a sheet of paper 1 foot long and 9 inches wide be divided? (Be careful not to use two different kinds of units. Change the length to inches before using the formula.)



9. The rectangles A , B , and C are drawn on "squared" paper. If each square represents one square inch find the area represented by each rectangle.

10. How many tiles each $1''$ square would be needed for the floor of a bathroom $10' \times 6'$? (First express $10'$ and $6'$ in inches.)

11. Check or prove the preceding example by first finding the area in square feet and the number of tiles needed for 1 square foot.

12. The dimensions of a sheet of metal are $8'$ by $28''$. How many square feet does it cover? (Express both dimensions in feet.)

13. At \$3 a square yard what would be the cost of a concrete walk $4'$ wide and $60'$ long?

14. In algebra the letter x is used so often that a product like $l \times w$ is apt to be confused with lxw . A different symbol for multiplication is needed, and we use a dot placed above the line, $l \cdot w$. (Note that the dot is not on the line as is a period or a decimal point.) Between two letters (or a number and a letter) we may omit the dot, so that the

three forms $l \times w$, $l \cdot w$, and lw all indicate multiplication and mean exactly the same thing. Calculate the value of $l \cdot w$ if $l = \frac{3}{4}$ and $w = \frac{2}{3}$.

15. What is the value of lw if $l = 5.5$ and $w = 2$?

16. Find the value of each product:

(a) $5 \cdot 3$

(c) $1000 \cdot 23$

(b) $100 \cdot 17$

(d) $\frac{1}{10} \cdot 25$

17. If $l = 6$ and $w = 5$ find the value of each of the following products:

(a) lw

(b) $7l$

(c) $100w$

18. Instead of "length" and "width" of a rectangle we frequently use "base" and "altitude." Write a formula for the area (A) of a rectangle in which you use b for the base and h for the altitude.

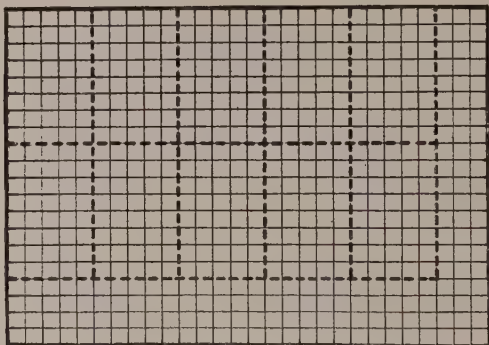
19. Using this formula, find the area of a rectangle whose base measures $16''$ and altitude $12''$.

Using the same formula, copy and fill in the blank space in each example below:

	BASE (b)	ALTITUDE (h)	AREA (A)
20.	$16''$	$9''$	
21.	$4'$	$2\frac{1}{4}'$	
22.	100 meters	35 meters	
23.	$15.4''$	$10''$	
24.	l	w	
25.	s	s	
26.	$16''$		80 sq. in.
27.		$6''$	1 sq. ft.

28. How many sheets of paper $5'' \times 8''$ could be cut from a sheet $20'' \times 28''$? Study the diagram, and find out by

sketching a similar diagram whether more sheets could be obtained by running them lengthwise.



*29. How many panes of window glass $9'' \times 12''$ could be cut from a piece of glass $4'$ long, $2' 3''$ wide? Make a sketch showing by dotted lines how the cuts could be made. What difference, if any, does it make whether the panes are cut across the large piece or lengthwise?

*30. A printer wishes to cut $3'' \times 5''$ cards, from sheets $20'' \times 28''$. How many square inches are wasted from each sheet if the cards are cut (a) lengthwise? (b) across? Do not try to answer until you have made two sketches, best done on squared paper (28 spaces $\times$ 20 spaces).

PERIMETERS

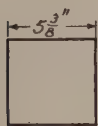
Many words in mathematics are easily understood if one knows the Latin or Greek languages in which these words were first formed. "Meter" means *measure* and is a part of such words as *thermometer* ("heat-measure"), *cyclometer*, *speedometer*, etc. "Peri" means *around*. Therefore by the

*Supplementary or extra credit examples are starred.

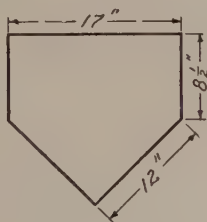
perimeter of a figure we mean the sum of the lengths of all its sides, or as we commonly say, "the distance around it."

EXERCISE 6

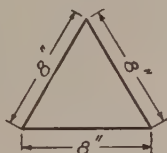
1. What is the perimeter of a square 12" on a side?
2. A square is $5\frac{1}{2}$ yards on a side. What is its perimeter?
3. A baseball "diamond" is really a square 90' on a side. What is its perimeter in feet? in yards?
4. Complete the following rule: "The perimeter of a square is equal to — times the length — — —."
5. Abbreviate this rule into a formula, using P to represent the perimeter and s to represent the side.
6. Use the formula from Example 5 to find P if $s = 36''$.
7. How long is a fence which surrounds a farm $\frac{1}{2}$ mile square?
8. Find the perimeter of each figure:



A SQUARE



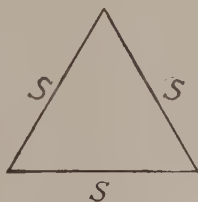
HOME PLATE



A TRIANGLE

9. Write a formula for the perimeter (p) of a triangle each side of which is s inches.

10. In $3s$, $3a^2$, or $3lw$ the 3 is called a numerical coefficient, or simply a coefficient. (a) Name the coefficient in $2l$, in $4bh$, and in $6s^2$.



(b) What coefficient is understood with a , t^2 , or mn ? (The coefficient 1 is often not written.)

(c) Complete: " $x + x + x = ?x$." The coefficient of the sum is ———?

11. Use coefficients to express more briefly each of the following:

$$(a) s + s + s + s$$

$$(c) lw + lw$$

$$(b) n^2 + n^2 + n^2$$

$$(d) b + b + h + h$$

12. What is the perimeter of a rectangle which is 22 meters long and 8 meters wide?

13. Copy and complete the rule: "The ——— of a rectangle is equal to twice its ——— plus ——— its ———."

14. Abbreviate this rule into a formula, using the letters P , l , and w .

15. Write a formula for the perimeter (P) of a rectangle in terms of its base (b) and altitude (h).

16. What is the perimeter of a rectangular city lot 125' deep and 34.06' wide?

$$\text{Solution: } l = 125'$$

$$w = 34.06'$$

$$P = ?$$

$$P = 2l + 2w$$

$$= 2(125) + 2(34.06)$$

$$= 250 + 68.12$$

$$= 318.12' \quad \text{Ans.}$$

Work out each of the following examples similarly:

17. What is the perimeter of a rectangle 28'' long and 22'' wide?

18. Find the perimeter of each of the rectangles A , B , and C on page 8.

19. A lot owned by Mr. Clark is $150' \times 100'$. What is the total length of the hedge which bounds Mr. Clark's lot?

20. What is the length of the fringe border needed to go around a rug 8' 6'' by 10'?

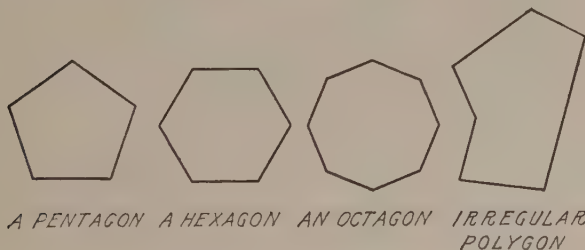
21. A football field is 100 yards long and 160 feet wide. What is its perimeter?

Find the perimeter of the rectangle in each of the examples below:

	LENGTH (l)	WIDTH (w)	PERIMETER (P)
22.	36"	1"	
23.	7' 3"	5' 3"	
24.	.78"	.37"	
25.	3'	18"	
26.	2 w	w	

27. What is the perimeter of a rectangle whose base is 36", if the altitude is one-half as long as the base?

28. Find P if $h = 9\frac{1}{2}$ inches and $b = 2h$.



29. A polygon with six sides is called a hexagon, one with eight sides an octagon. What would be the perimeter of an octagon if each of its sides is 25' long?

30. Write a formula for the perimeter (P) of a hexagon if its sides are all of the same length (s).

31. Find the perimeter of an irregular polygon of five sides measured by a surveyor as follows: 127.34', 205.10', 67.63', 130.79', and 40.08'.

THE MEASURING OF LINES

In the previous pages the dimensions of the figures have been given to you or else you could easily determine them by observation of a diagram. In practical work it is very often necessary to measure certain lines in order to find perimeters and areas.

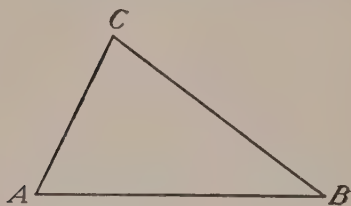
To measure straight lines we use

- (1) A rule or scale marked in inches and fractions of an inch or in centimeters and millimeters.
- (2) A steel or cloth tape, especially for out of doors.
- (3) A pair of dividers or compasses.
- (4) Squared paper.

EXERCISE 7

1. Measure the length of a page of this book as accurately as possible.

2. With a scale, measure to the nearest $\frac{1}{16}$ " the three sides of the triangle ABC and find its perimeter.



$AB =$

$BC =$

$CA =$ _____

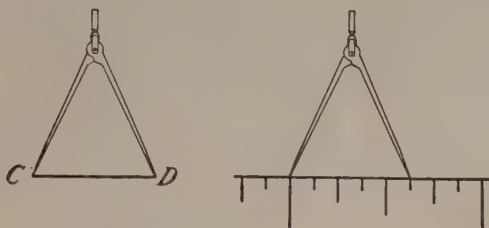
Perimeter =

3. Find to the nearest $\frac{1}{16}$ " the perimeter of the hexagon on page 13. Arrange the work as in the example above.

The Use of Dividers and Compasses in the Measuring of Lines

To measure a line CD with the dividers, place one point of the dividers at C and open the dividers just far enough so that you can place the other point at D . Without

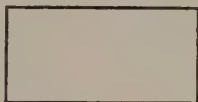
changing the opening of the dividers, place one point of the dividers at some mark on your ruler and observe where



the other point falls. Thus in the diagram above the length of CD is $\frac{5}{8}$ ".

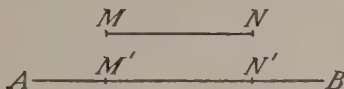
EXERCISE 8

1. Using the dividers and scale, measure the length and the width of the rectangle and calculate its perimeter. What formula will you use for the perimeter?



By the use of the dividers, or compasses, it is possible to lay off a line equal in length to a given line, without measuring with a ruler.

To lay off from M' on AB a length equal to MN , first place the two points of the compasses on M and N ; then

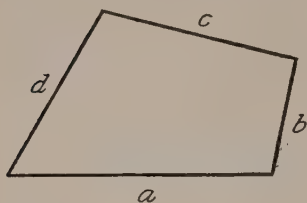


with one of the points at M' mark the position of the other on AB at N' . $M'N'$ will be equal to MN . (M' is to be read: " M prime.")

2. With compasses mark off on a long line XY the lengths of the four sides of the rectangle in Example 1, and with the

ruler measure the total length. Compare your answer with the answer to Example 1.

3. Measure each of the four sides of the quadrilateral (four-sided figure) given here and compute:



$$a =$$

$$b =$$

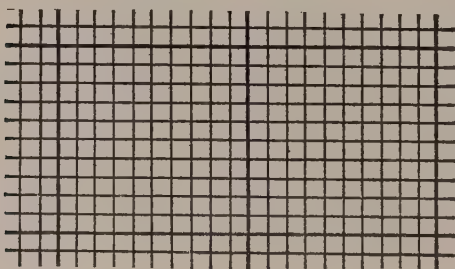
$$c =$$

$$d =$$

$$\text{Perimeter} = \underline{\hspace{2cm}}$$

4. Mark off on a long line, with compasses, the lengths of a , b , c , and d , and with a ruler measure the total length. Compare with Example 3.

Squared Paper



In the squared paper above, the large squares are 1" on a side. The small squares are all equal. What is the length of the side of a small square?

With squared paper we can measure the length of a line to $\frac{1}{16}$ " and can estimate with a little practice to $\frac{1}{100}$ ". Thus, in the diagram on opposite page the line AB is about 1.53".

The point B falls between the fifth and sixth subdivisions,

thus making the line AB between 1.5'' and 1.6''; dividing the side of the small square with the eye into ten equal parts,



we estimate that point B falls on the third division point, making the line AB 1.53'' in length.

EXERCISE 9

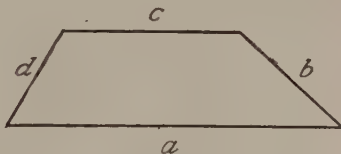
1. Using compasses and squared paper, find the lengths of the sides of the quadrilateral below and calculate the perimeter:

$a =$

$b =$

$c =$

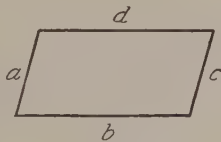
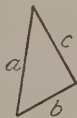
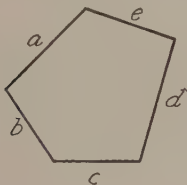
$d =$



Perimeter = _____

2. With compasses lay off in one long line, on squared paper, lines equal to a , b , c , and d . What is the combined length? Compare with Example 1.

3. Find the perimeter of each of the three figures below: first, by finding the length of each side as in Example 1; and second, as a check, by laying off the lengths of the sides in one line, as in Example 2.



THE LANGUAGE OF ALGEBRA

The abbreviations and symbols of algebra greatly simplify and speed up mathematical work, just as shorthand abbreviates ordinary words and speeds up writing. One may think of the language of algebra as a kind of cipher or code by means of which one can express the meaning of a long message by a few signs and letters. The code-book or vocabulary of algebra consists chiefly of (1) letters, (2) the signs of arithmetic, (3) exponents, (4) coefficients. These are combined to form terms, equations, formulas, etc., which are like phrases and sentences.

(1) **Letters.** From the preceding pages you are already familiar with the use of letters to represent various quantities, such as s for the length of a side, P for perimeter, etc. In different examples a letter may not only have different values but may represent different quantities. As the side of a square, s may be 100'' in one case and $5\frac{1}{2}'$ in another. In one example, r may represent the radius of a circle, and in another r may be used for rate of interest.

(2) **Signs and Words of Arithmetic.** Nearly all the signs and words of arithmetic are used also in algebra, with the same meanings.

What is the value of $7 + 5$? of $8 - 3$? of 9×6 ? of $100 \div 4$?

The first example may be expressed, "What is the sum of 7 and 5?" or "Add 7 and 5." Express in two different wordings each of the other three examples.

The result of the first example may be stated as a sentence, "The sum of 7 and 5 is 12." Make three sentences based on the results of the other examples, each sentence to contain one of the words, "product," "quotient," or "difference."

(3) **Exponents.** Exponents and powers have been explained on page 5.

(4) **Coefficients.** In the formula $p = 4s$ the quantity $4s$ means $4 \times s$. If $s = 12$, what is the value of $4s$? The coefficient is always a multiplier and indicates that there is a product. $16t^2$ means $16 \times t^2$, the coefficient being 16.

Exponents and coefficients enable us to save time and space in writing numbers and in calculating with them. Instead of writing $a + a + a$ we write $3a$, just as instead of $s \times s$ we write s^2 .

What is the numerical coefficient in $4s^3$? What is the exponent?

How can you write more compactly $x + x + x + x$?

In what way can $m \cdot m \cdot m$ be written in a briefer form?

(5) **Terms.** In such an expression as $3a^2 + 2b$ there are said to be two terms: the first term is $3a^2$ and the second term $2b$. Terms are separated from each other by $+$ or $-$ signs.

How many terms are there in $5 + lwh + 6s^2 - d^3$?

Write an expression of three terms.

(6) **Equations and Formulas.** An equation or formula is a statement that two expressions are equal, each expression being made up of various terms, signs, letters, etc. As suggested above, equations and formulas correspond to the sentences of everyday language which are made up of various words and punctuation marks. The equation $7 + 5 = 12$ means the same as the sentence "The sum of 7 and 5 is 12." The formula for the area of a rectangle, $A = lw$, means the same as what sentence? In the next few pages you will make the acquaintance of many formulas and equations.

Just as one cannot master all of a foreign language (or of a code) in a single day, so you cannot learn the language of algebra in one lesson. Day by day you will find out new things about this language and its wide usefulness.

EXERCISE 10

With the aid of symbols write each of the following in the shortest way possible:

1. $a \cdot a$.
2. The product of l and w .
3. The sum of 7 and 6.
4. $t + t + t + t$.
5. 18 divided by 5.
6. a divided by b .
7. The sum of $2l$ and $2w$.
8. a plus b plus c .
9. 7 subtracted from 20.
10. The difference of a and b .
11. The product of l , w , and h .
12. The cube of e .
13. The square of s plus the square of s .
14. $r + r + s + s + s = (?) r + (?) s$.
15. $a + a + a + a + b + b$.
16. $D \cdot D \cdot D$.
17. $6 \cdot a \cdot a$.
18. $c \cdot h \cdot h \cdot h$.
19. $5 \cdot b \cdot b \cdot b \cdot c \cdot c$.
20. $h \cdot h + R \cdot R \cdot R = h^? + R^?$

Each of the next eight should be expressed as an equation:

21. The sum of 13 and 12 equals 25.
22. The sum of $5m$ and $4m$ is $9m$.
23. The product of $\frac{3}{4}$ and $\frac{1}{2}$ is $\frac{3}{8}$.
24. A equals the product of b and h .
25. The difference of 19 and 12 is equal to 7.
26. The difference of x and y is equal to d .
27. The quotient of 100 and 8 is equal to $12\frac{1}{2}$.
28. The quotient of 8 and 100 is $\frac{2}{25}$.

29. Write as a single term: $n^2 + n^2 + n^2$.
30. Write as the sum of two terms: $x^3 + x^3 + y + y + y$.
31. Find the value of $7t$ when $t = 12$; when $t = 100$.
32. Find the value of a^2 and of $2a$ when $a = 5$.
33. What is the value of s^3 and of $3s$ if $s = 2$?
34. When $x = 10$, what is the value of x^2 ? of $3x^2$? of $2x^3$?
35. Name the coefficient and the exponent in x^2 , in $3x^2$, and in $2x^3$.
36. How many 10's are multiplied together to make 1000? Write 1000 as a power of 10.
37. Write 25 as a power of 5; 16 as a power of 2.

If $a = 1$, $c = 2$, and $x = 5$, find the value of each of the following:

- | | |
|-----------------------|-----------------------------------|
| 38. $x + 4$ | 49. $\frac{x - c^2}{a}$ |
| 39. $4x$ | 50. $ac^2 - a^2c$ |
| 40. $a + c + x$ | 51. $cx^3 - 100a$ |
| 41. acx | 52. $\frac{1}{8}x + \frac{1}{2}c$ |
| 42. $x - 2c$ | 53. $c^3 - 3c$ |
| 43. c^3 | 54. $2x + x^2$ |
| 44. $10a + cx$ | 55. $\frac{x}{c} - \frac{5}{x}$ |
| 45. $x^2 - a^2$ | 56. $a^4 + 4a$ |
| 46. $5c^2$ | |
| 47. $2x^2$ | |
| 48. $\frac{a + x}{c}$ | |

In the next three examples use the values of a , c , and x as given above.

57. If $y = a + x - c$, what is the value of y ?
58. Find the value of m if $m = a^2 + c^2 + x^2$.
59. Find the value of m if $m = a^2c^2x^2$.
60. Find the value of p if $p = c + c^2 + c^3$.

FORMULAS

It is often important to know how to obtain a certain unknown number from other numbers which are known. The value of the number sought depends upon the values of the known numbers and is in some definite way related to them. Thus the area of a rectangle depends upon the values of two numbers, the length and the width. The relation between the area and these two numbers is definitely stated by the formula $A = lw$.

There are many ways of expressing relationship between numbers. In arithmetic it is usually expressed by a rule stated in words; by using the language of algebra we abbreviate these rules into *formulas*. When you are trying to get the formula which will correspond to a word statement, it is a good plan to write the words on a single line if possible and then place directly beneath each word or phrase the algebraic symbol that has the same meaning:

Example 1.

The perimeter of a square is four times the side,

$$\begin{array}{ccccccc} P & & = & 4 & \times & & s \\ \text{or} & & & P = 4s & & & \end{array}$$

Example 2.

The selling price equals the cost plus 10% of the cost,

$$\begin{array}{ccccccc} s & & = & & c & + & \frac{1}{10}c \\ \text{or} & & & s = c + \frac{1}{10}c & & & \end{array}$$

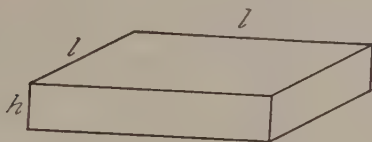
Example 3.

The time required equals the distance traveled divided by the rate.

$$\begin{array}{ccccccc} t & & = & & d & \div & r \\ \text{or} & & & t = \frac{d}{r} & & & \end{array}$$

Example 4.

The surface of a square box is equal to the sum of twice
 S $=$ 2
the square of its length and four times the product of the
 l^2 $+$ 4 $\times$
length and the depth,
 l h
or $S = 2l^2 + 4lh.$



The most common relations between numbers are those obtained by the four fundamental operations: addition, subtraction, multiplication, and division.

EXERCISE 11

The Product Relation. From your study of the rectangle you are familiar with the kind of quantity which is obtained by taking the product of two numbers. Such quantities and such product formulas occur frequently.

1. Write a formula for the area of a rectangle, representing the length by x and the width by y .

2. What would be the cost of 3 apples if the price is 6 cents apiece?

3. What would be the cost of x apples if the price is y cents apiece?

4. Write a formula giving the cost (C) of any number of apples (n) if the price of one is p cents.

5. What would 6 yards of cloth cost at 15 cents per yard?

6. How much would 6 yards of cloth cost at m cents per yard?

7. Write a formula for the cost (C) of any number of yards (y) of cloth at p cents a yard.

8. "The diameter of a circle is twice as long as the radius." Translate this rule into a formula, using D and R .

9. Translate in a similar way: "The radius of a circle is one-half as long as the diameter."

10. The parcel post rate for sending a package to the 8th "zone" (any distance over 1800 miles) is 12 cents for each pound. What would be the postage to this zone on a parcel weighing 3 pounds? On one weighing 7 pounds?

11. Write a formula for the postage (p) to the 8th zone for a parcel weighing w pounds.

12. How many inches are there in 5 feet? in x feet?

13. Write a sentence giving a rule in words for changing any number of feet to inches.

14. Write a formula for the number of inches (i) in any number of feet (f).

15. Write a similar formula for the number of square inches (I) in any number of square feet (F).

16. How many cubic feet are contained in a steam-shovel scoop which holds 2 cubic yards?

17. Give a formula for the number of cubic feet (f) in any number of cubic yards (y).

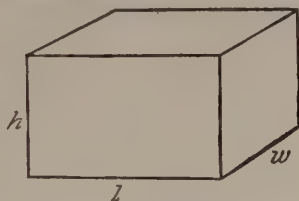
18. How far will an aeroplane go in 10 hours if it maintains a speed of 165 miles an hour?

19. How many miles can a scout hike in a day if he can walk for 8 hours at the rate of $3\frac{1}{2}$ miles per hour?

20. State a formula for the distance (d miles) an automobile travels in t hours at the rate of r miles per hour.

21. Translate into a formula: "The volume of a rectangular solid is equal to the product of its three dimensions" (l , w , h).

22. A school room is 25' long and 12' high. How many square feet in each of the side walls?



23. The school room is 20' wide. What is the area of the end walls? Of all four walls? Of the ceiling?

24. A box measures 20'' long, 12'' wide, and 10'' high. Find the area of (a) the two ends; (b) the two sides; (c) the top and bottom.

25. A room is l' long, b' broad, and h' high. What is (a) the area of the floor; (b) the area of the two ends; (c) the area of the sides?

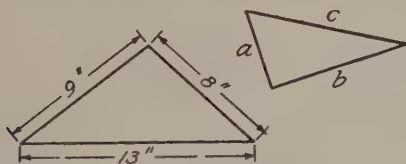
EXERCISE 12

The Sum Relation

1. How can you indicate the sum of the two numbers 7 and 3?

2. Express similarly the sum of the two numbers x and y .

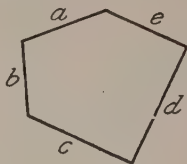
3. How can you state in the form of an equation that the sum of the three numbers 25, 29 and 32 is 86?



4. What is the perimeter of the triangle above at the left?

5. Express a formula for the perimeter (P) of the second triangle above (sides a , b , and c).

6. Express a formula for the perimeter of an equilateral triangle, each side being a .



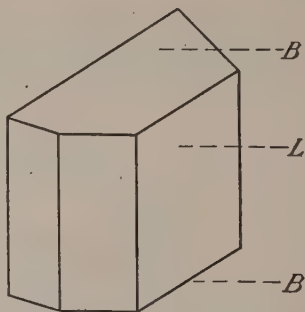
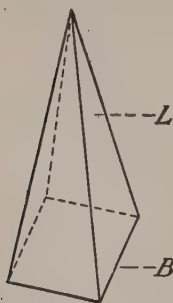
7. The first triangle above has two equal sides and is called an isosceles triangle. State a formula for its perimeter.

8. Write a formula for the perimeter of the five-sided polygon.

9. Your father is 35 years older than you are. How old will he be when you are 17 years old? 25 years old?

10. If your father is 35 years older than you are, write a formula for his age (f) in terms of your age (a).

11. Translate into a formula: "The total surface (S) of a pyramid is equal to the sum of the lateral surface (L) and the base (B)."

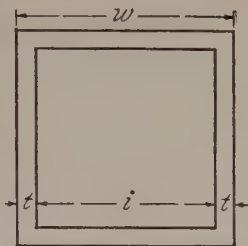


12. Write a similar formula for the total surface of a prism. (Use S , L , and B . A prism has two equal bases.)

13. A wooden box is i inches square inside and the thickness of the wood is t inches.

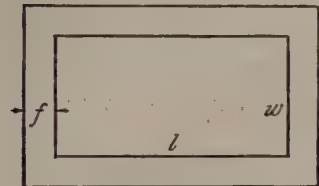
(a) Write a formula for the outside dimension w .

(b) What would be the value of w if $i = 10''$ and the boards are $1''$ thick?



14. The printing on each page of a book is to fill a space $6'' \times 4''$ and is surrounded by a margin $\frac{3}{4}''$ wide. What will be the outside dimensions of a page?

15. A picture $l'' \times w''$ is surrounded by a frame f inches wide.



(a) Express a formula for the outside length L in terms of l and f .

(b) Express a formula for the outside width W in terms of w and f .

(c) Show that the outside perimeter of the frame is $2l + 2w + 8f$.

16. In buying a piano on the installment plan Mr. Jones agrees to pay \$25 down and \$3 a week. How much will he have paid in 5 weeks? in 52 weeks?

17. Write a formula for the amount (A) Mr. Jones will have paid in w weeks.

EXERCISE 13

The Difference Relation

1. (a) How can you indicate the difference between 23 and 15?

(b) How can you indicate the difference between a and b ?

2. How can you state by an equation that the difference between $\frac{3}{4}$ and $\frac{2}{3}$ is $\frac{1}{12}$? that the difference between a and b is c ?

3. If your age is 16 years how old were you 7 years ago?

4. What was your age n years ago if you are now a years old?

5. An empty auto truck weighs 2500 lb. After loading with coal it weighs 7000 lb. What is the weight of the coal?

6. An empty butter firkin weighs $4\frac{3}{4}$ lb. When filled with butter it weighs $65\frac{1}{2}$ lb. How many pounds of butter are there?

7. Express a formula for the net weight (w) if the weight of the container (known as "tare") is t and the combined or gross weight is g .

8. "To find a golfer's net score you subtract his handicap from his actual score." Express this rule as a formula (n, h, a).

9. In a 440-yard handicap race a contestant does not run the full distance, having been granted a head start or handicap of h yards. Write a formula for the distance (d) which he actually runs.

10. A steamer which has a speed of 15 miles per hour in still water can make only 11 miles an hour against the current of a certain river. What is the rate of the flow of the current? At what rate of speed could the steamer go down the river?

11. A steamer has a speed of s miles per hour in still water. Express formulas for its actual rate of progress up (u) a river and down (d) a river if the current flows at the rate of 4 miles an hour. Write also formulas for u and d if the rate of the current is c miles per hour.

12. How do you find the selling price of an article if you know the cost and the gain in dollars? Express in symbols (s, c, g).

13. Express a similar formula for the selling price when the article is sold at a loss.

14. Write a formula for finding the gain in terms of the selling price and the cost.

15. A dealer in candy advertises to sell candy at "a penny a pound profit." Write a formula for the selling price per pound in terms of the cost.

16. If Frank had b dollars in the savings bank and withdrew w dollars, write a formula for his balance (B) today.

17. Express a formula for a man's bank balance (B) today if his balance yesterday was b dollars and if during the day he deposited d dollars and the bank paid out c dollars on his checks.

18. Mr. Lynn owed a debt of \$200 which he promised to pay back at the rate of \$15 a month. What amount will he still owe after making the payments for 5 months? for 1 year?

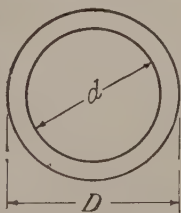
19. Write a formula for the amount of Mr. Lynn's debt (d) after m months.

20. The outside diameter of a piece of pipe is 24" and the thickness of the pipe is $\frac{1}{2}$ ". What is the inside diameter?

21. Write a formula for the inside diameter (d) if the outside diameter is D and the thickness of the pipe is t .

*22. There are 600 cubic centimeters of water in a measuring glass. An electric-light bulb is immersed in the water, which then rises to the mark for 880 cubic centimeters. What is the volume of the bulb in cubic centimeters?

*23. Express a formula for V in the preceding example, if there are w cubic centimeters of water in the glass and the water rises to the mark for w' (read it " w -prime") cubic centimeters.



EXERCISE 14

The Quotient Relation

1. How can you indicate the quotient of 85 and 13? Write it also as a fraction.
2. Write as a fraction the quotient of a and b .
3. State by a formula: "The number of pounds (n) of candy one can buy for 90 cents is the quotient of 90 and the price per pound (p)."
4. The area of a rectangle is 120 sq. meters and the length is 15 meters. What is the width?
5. Write a formula for the width (w) of a rectangle in terms of A and l .
6. Write a formula for the base of a rectangle in terms of the altitude and the area (b , h , A).
7. Express a formula for the number (n) of five-cent pencils one can buy for c cents.
8. Give a formula for changing inches to feet.
9. Write a formula for changing ounces to pounds.
10. A good way to check division examples is by multiplication. Write a formula for the dividend (D) if you know the divisor (d) and the quotient (q).
11. Write a formula for the divisor (d) if the dividend is D and the quotient is q .
12. Copy and complete: "In dividing 87 by 13 the quotient is — and the remainder is —."
13. Tell what was the dividend if the divisor was 15, the quotient 7, and the remainder 11.
14. Write a formula for the dividend (D) in terms of the quotient (q), the divisor (d), and the remainder (r).
15. Write a formula for the area of a square (A) in terms of the length of a side (s).
16. Write a formula for the volume of a cube (V , e).

17. What would be the total surface in square inches of a cube having its edges 3'' long? 1'' long?

18. Write a formula for the surface of a cube (S, e).

19. Translate into a formula: "The number of feet (d) passed over by a falling object is equal to 16 times the square of the number of seconds (t) during which it has fallen."

20. Translate into a formula: "The horse-power ($H.P.$) of an automobile engine is equal to the number of cylinders (N) multiplied by the square of their diameter (D) divided by 2.5."

21. Translate into a formula the interest rule: "The interest is found by multiplying the principal by the rate by the time in years." ($i, p, r, t.$)

22. Using the formula, find the interest on

(a) \$500 at 6% for 3 years.

(b) \$1250 at 4% for $1\frac{1}{2}$ years.

23. What fraction of a year (use 360 days) is 30 days? 1 day? 47 days? n days?

24. Write a formula giving the interest on p dollars at r per cent for n days.

25. Using this formula, find the interest on

(a) \$60 at 7% for 19 days.

(b) \$3250 at $5\frac{1}{2}\%$ for 10 days.

ORDER OF OPERATIONS

When we are *adding* numbers it makes no difference in what order the numbers are taken. Indeed the best way to check the addition of a column of figures is to add in the reverse order: if you have already added up the column then check by adding down. Sometimes the numbers to be added are such that it is more convenient to add in some particular order, perhaps an unusual one, than in any other. (See Example 1, of Exercise 15.)

Numbers may also be *multiplied* in any order, and a good check for multiplying 378 by 597 is to multiply 597 by 378.

When an example calls for additions and also subtractions we may perform the operations in any order we like. $12 - 20 + 18$ is the same as $12 + 18 - 20$; $x^2 - 5 + 3x$ may be written $x^2 + 3x - 5$.

In a more complicated example like $5 + 2 \times 7 - 12 \div 6$ the multiplications and divisions are to be performed before the additions and subtractions:

$$\begin{array}{r r r r r r r} 5 + 2 \times 7 - 12 \div 6 = \\ 5 + 14 - 2 = \\ 19 - 2 = 17 \end{array}$$

To find the value of such an expression as $5a^2 + 2ab - a^3$ when $a = 3$, $b = 4$, the value of each term is first obtained separately:

$$\begin{array}{r} 5a^2 + 2ab - a^3 = \\ 5 \cdot 3^2 + 2 \cdot 3 \cdot 4 - 3^3 = \\ 5 \cdot 9 + 2 \cdot 12 - 27 = \\ 45 + 24 - 27 = \\ 69 - 27 = 42 \end{array}$$

The parenthesis in $4(5 + 3)$ is a symbol of grouping, and the expression within the parenthesis must be handled as a single number. In any example, the operations inside the parenthesis are to be done first of all.

EXERCISE 15

1. Add mentally each of the following, taking the numbers in the easiest order:

- (a) $5\frac{3}{8} + 4\frac{5}{8} + 7\frac{1}{2}$
- (b) $3\frac{2}{7} + 1\frac{5}{16} + \frac{5}{7}$
- (c) $2 + 3 + 7 + 4 + 6$
- (d) $\$2.78 + \$1.67 + \$.33$

2. If a and b represent any two numbers state in words what is meant by $a + b = b + a$ and by $ab = ba$.

3. Set down in a column all the odd numbers from 1 to 29. Add them down the column, then cover your answer with a slip of paper and add up the column.

4. Repeat Example 3 with all the even numbers from 80 to 118.

5. Multiply each of the following mentally, taking the numbers in the easiest order:

(a) $\frac{1}{2} \times 173 \times 20$

(b) $97 \times 4 \times 25$

(c) $16\frac{2}{3} \times 23 \times 6$

Find the value of

6. $10 \times 12 - 5$

11. $8(5 + 6)$

7. $17 + 36 \div 4$

12. $13 - 7 - 2$

8. $3 \times 7 - 7 \times 3$

13. $13 - (7 + 2)$

9. $4^2 - 10 \div 2$

14. $5 + 3(12 - 8)$

10. $48 \div 3 + 20 \div 5$

15. $(5 + 3)(12 - 8)$

Find the value of the following when $r = 2$, $s = 3$, $t = 4$, $x = 0$:

16. $r + s^2$

25. $(s - r)^2$

17. $rs + st$

26. $s^2 + r^2 - 2sr$

18. $t + r(t - r)$

27. $r + s + t + x$

19. $(t + r)(t - r)$

28. $x + x + x$

20. $(t + r)t - r$

29. $5x$

21. $(r + s)^2$

30. tx

22. $r^2 + 2rs + s^2$

31. $t - x$

23. $\frac{t^2 - s^2}{t + s}$

32. $r - s^2 + t^3$

24. $3r^2 - (2t + s)$

33. $rst - 24$

If $l = 5$, $w = 2$, $h = 4$, $k = 1$, what is the value of:

34. $\frac{l}{w} + \frac{k}{h}$

39. $3l^2 + 2k^3$

35. $2lw + 2wh + 2lh$

40. $2l^3 - 3h^2$

36. $\frac{l}{h} + \frac{l}{k} + \frac{l}{w}$

41. $(l + w + h)k$

37. $l^3 + k^3$

42. $\frac{lh - w^2}{wh}$

38. $\frac{h}{w} + \frac{k}{l}$

43. $\frac{1}{2}lw + \frac{1}{3}(l - w)$

CHAPTER II

GEOMETRIC CONSTRUCTIONS AND AREAS

THE USE OF COMPASSES

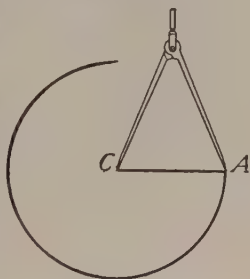
Construction Problem 1. *How to draw a circle with a given radius.*

Step 1. Draw a line CA one inch long.

Step 2. Place the metal point of the compasses at C .

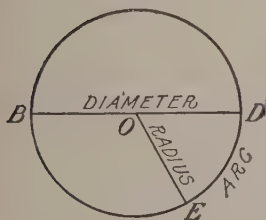
Step 3. Adjust the opening of the compasses until the pencil point is on A .

Step 4. Keeping the metal point fixed at C , turn the other arm of the compasses about the point C , without changing the opening of the compasses and without lifting the pencil point from the paper. The pencil point will draw a circle.



Draw several lines from the center C to different points in your circle. What is the length of each of these lines? Each of the lines is a radius of the circle. What is true of the lengths of all the radii of the same circle?

EXERCISE 16



1. In the figure, OB , OD , and OE are radii, BD is a diameter, and DE is an arc. Can you name another arc?

2. The diameter of a circle passes through the center. How long is the diameter of the circle which you drew in Problem 1?

3. How can you find the diameter of any circle if you know the radius? What is the diameter of a circle whose radius is 7''?

4. Write a formula for the diameter (D) of a circle in terms of the radius (R).

5. Draw a circle with radius 2''. With the same center draw another circle with radius $1\frac{1}{2}$ '. (Circles which have the same center are called concentric circles.)

6. Draw a line AB 4'' long. With A and B as centers and with a 2'' radius draw two equal circles. (The circles should touch, and are called tangent circles.)

7. Draw a line CD 2'' long. With C as center draw a circle having a 3'' radius; with D as center draw a circle having a 1'' radius.

8. Draw a line PQ 2'' long. Mark any point on the line M . Draw four circles with center and radius as follows:

	Center	Radius
(a)	P	PM
(b)	P	PQ
(c)	Q	QM
(d)	Q	QP

9. Draw a line BC $1\frac{3}{4}$ ' long. With B as center and radius equal to BC draw a circle. Draw an equal circle with center at C , and call one of the points where the circles intersect, A . Draw the lines AB and AC , forming a triangle. What is true of the lengths of the lines AB , AC , and BC ? Such a triangle is called an equilateral triangle.

10. Draw a line BC $2\frac{3}{8}$ ' long and try to construct an equilateral triangle, drawing only short arcs to get the intersection point A .

11. Draw a line BC 2'' long. Draw circles from B and C as centers each with radius $2\frac{1}{2}$ ', intersecting at A . Draw

AB and AC ; how do their lengths compare? What kind of a triangle is ABC ? See page 26.

12. Repeat Problem 11, making BC $2\frac{1}{2}$ " long and the other sides each 2" long, using only short arcs to find A .

13. Draw a line BC 3" long. Draw a circle from B as center with a 2" radius, and another circle from C as center with a $2\frac{1}{2}$ " radius. If A is an intersection point of the two circles draw the triangle ABC . What are the lengths of the three sides of the triangle BC , BA , and CA ?

14. Without drawing complete circles but merely short arcs construct a triangle ABC having $BC = 2\frac{3}{4}$ ", $BA = 2$ ", and $CA = 1\frac{1}{2}$ ".

15. Construct a triangle ABC , having its three sides equal to these lines:

B ————— C
 B ————— A
 C ————— A

16. Construct an isosceles triangle having its base and its arm equal to these lines:

base
 —————
 arm
 —————

THE CIRCUMFERENCE OF A CIRCLE

The length of the curved line which forms a circle is called the circumference.

Around a wooden cylinder or a cylindrical tin can wrap a narrow strip of squared paper until the ends overlap. With a point of the dividers prick a hole through both ends of the strip where they overlap. Unroll the paper and lay it out flat. The distance between the two small holes is the circumference of the circle which forms the base of the cylinder. Measure the circumference, and measure as nearly as possible the diameter of the cylinder.

EXERCISE 17

1. Divide the circumference you have just found by the measured diameter. Carry out the division to two decimal places.

2. A student found the circumference of a cylinder to be $12\frac{9}{16}$ " , the diameter being 4". What was the quotient of the two (to the nearest hundredth)?

3. A bicycle wheel is 28" in diameter. By marking with chalk the point on the tire where it touched the ground, and rolling the bicycle along until the same point again touched the ground, a boy found the circumference to be 88". Divide 88 by 28, expressing your answer as a mixed number, and also to the nearest hundredth.

If you divide the circumference of any circle by its diameter the quotient will always be about 3.14. The value of the quotient cannot be expressed exactly either as a common fraction or as a decimal, but the approximations commonly used are $3\frac{1}{7}$, 3.14, and 3.1416. The quotient is used so frequently that we have a symbol for it, the Greek letter π (pronounced like the word *pie*).

4. Using $\pi = 3\frac{1}{7}$ calculate the circumference of a cylinder $3\frac{1}{2}$ " in diameter.

5. State a rule for finding the circumference of a circle when you know the diameter.

6. Write a formula for the circumference (C) of a circle in terms of π and the diameter (D).

7. Using the formula you have just written, find the circumference of a circle whose diameter is 21". Take $\pi = 3\frac{1}{7}$

8. In the formula for C replace D by its value in terms of R , and get a formula for the circumference in terms of the radius.

Find the circumference of each of the following circles:

(Use $\pi = 3\frac{1}{7}$)

(Use $\pi = 3.14$)

9. Diameter 20 meters

12. $D = 12''$

10. Radius $2\frac{1}{4}''$

13. $R = 1.73''$

11. $R = 4'2''$

14. $D = .075$ centimeters

(Use $\pi = 3.1416$)

15. $R = 1''$

16. $R = 5730'$

17. $D = 39.37''$

18. An automobile wheel is $35''$ in diameter; find its circumference ($\pi = 3\frac{1}{7}$). How far would the automobile move forward while the wheel turns exactly one revolution?

19. Change the last answer to feet and find how many times the wheel would revolve in going one mile, or $5280'$. (These examples suggest the basis for designing cyclometers and speedometers.)

20. A giant tree in California is $26'$ in diameter. What is the distance around the tree?

21. If men stand with outstretched arms, their fingertips just touching, each man can stretch about $6'$. How many men would it take to reach all the way around the tree?

22. How wide a piece of sheet iron would be needed to bend into a cylindrical stove pipe $7''$ in diameter?

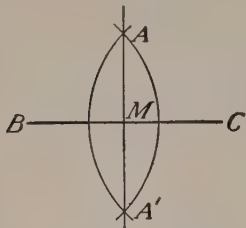
23. How long a piece of rubber would be used for the tire of a baby carriage wheel, $16''$ in diameter?

CONSTRUCTIONS

Construction Problem 2. *How to bisect a line.*

We have learned in Construction Problem 1 how to draw a circle when the radius is given. Suppose we are told to draw a circle having a certain line AB as the diameter; we cannot do it unless we know how to divide AB into two equal

parts (to "bisect" it). The following construction shows how to bisect the line BC .



Step 1. Using B as center and a radius estimated to be greater than $\frac{1}{2} BC$, draw an arc extending above and below BC .

Step 2. With C as center and the same radius as from B , draw an arc intersecting the other at A and A' .

Step 3. Join A with A' . Then AA' is the bisector, and BM is equal to MC .

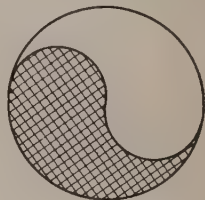
EXERCISE 18

1. Draw a line BC about 3" long. Bisect it carefully, as in the diagram above.

2. Bisect a line BC about 4" long, calling the middle point M . Bisect the line BM , calling the middle point M' . What fraction of the length of BC is BM' ?

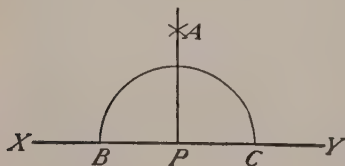
3. Given a line AB . Construct a circle that shall have AB as diameter.

4. Can you see how this design was constructed? Draw one yourself, making the diameter of the circle three times what it is here.



5. Draw any triangle. Bisect each of its sides and join the mid-points to form a smaller triangle.

Construction Problem 3. *How to erect a perpendicular to a line at a given point.*



Step 1. With the given point P as a center and with any radius, draw a semicircle. Let the points where the semicircle meets the given line, B and C .

Step 2. Using B as center with a radius longer than BP draw an arc above P .

Step 3. Using C as center and the same radius as from B , draw another arc, intersecting the first arc.

Step 4. Mark with the letter A the point where the arcs intersect. Join AP .

Then AP is the perpendicular required.

The two angles formed by the perpendicular at P are called *right angles*. If AP were prolonged below the line XY there would be four right angles at P . What kind of figure has four right angles at its four corners? What is the name of an equilateral figure which has right angles at its four corners?

The bisector of a line as drawn in Construction Problem 2 forms right angles with the given line and is said to be the perpendicular bisector of the line.

EXERCISE 19

1. Draw a line CD about 3" long. Letter any point in CD , A . At A erect a perpendicular to CD .

2. On a line EF about 5" long take two points G and H exactly 2" apart. At G and H erect GI and HJ perpendicular to EF . (The two lines GI and HJ would never meet, always remaining two inches apart, and are said to be parallel lines.)

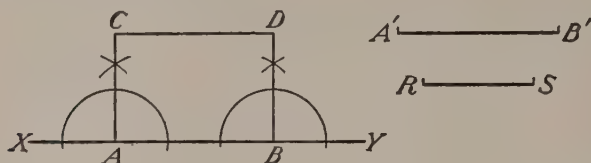
3. On a line about 6" long take two points A and B exactly 3" apart. Erect two perpendiculars AD and BC each 2" long. Draw CD . What kind of figure is $ABCD$?

Construction Problem 4. *How to construct a rectangle when the base and the altitude are given.*

Step 1. On a line of indefinite length, XY , lay off AB equal to $A'B'$, the base.

Step 2. At A and at B erect perpendiculars.

Step 3. On each of the perpendiculars lay off the length RS , getting AC and BD .



Step 4. Join CD . $ABDC$ is the rectangle required.

Step 5. Test by measuring CD as compared to AB .

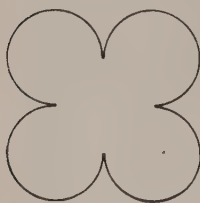
EXERCISE 20

1. Draw two lines and construct a rectangle having its base and altitude equal to them. Test.

2. In the diagram for Construction Problem 4 what three different radii were used with A as center? Can you count eight times when the compasses were placed?

3. Construct a rectangle with base $2\frac{1}{2}$ " and altitude $1\frac{1}{2}$ ". Measure the two diagonals and compare them. (In all these

constructions leave all construction arcs and lines on your paper.)

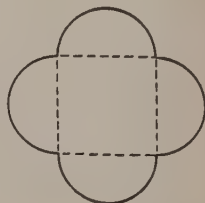


4. To construct a square, given its side, proceed just as for the rectangle, using the given length for the altitude as well as the base. Construct a square with sides $2\frac{1}{2}$ " long.

5. Construct a square with sides equal to a given line RS .

6. Can you see how the design above was constructed? Draw one like it yourself.

7. A famous design consists of four semicircles constructed outward on the



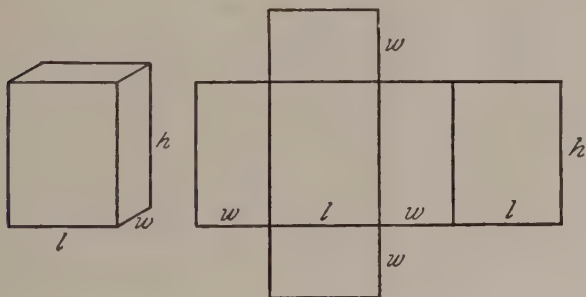
sides of a square. Make such a design, constructing it carefully, based on a square with its side equal to AD ,

A ————— D .

(Hint: See Example 3 of Exercise 18.)

8. Another design consists of four semicircles constructed inward on the sides of a square. Construct it on a 3" square.

9. A running track is based on a rectangle plus two semicircles constructed on the ends of the rectangles as diameters. Construct a plan for such a track based on a rectangle 3" x $2\frac{1}{2}$ ".



10. Construct on heavy paper or cardboard a pattern for a rectangular solid with $l = 1\frac{1}{2}$ ", $w = 1$ ", $h = 2$ ". First carefully construct a rectangle $1\frac{1}{2}$ " x 2". After completing the pattern cut it out and fold to form a solid, pasting strips of paper along the edges if needed.

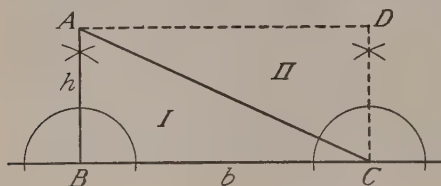
11. How many square inches are there in the entire surface of the solid in Example 10?

12. Make up a formula for the surface (S) of a rectangular solid (l, w, h).

13. Construct from cardboard a 2" cube, first making with ruler and compasses a pattern similar to that above.

TRIANGLES

A perpendicular is erected at B ; any point in the perpendicular A is joined to C , forming a triangle with a right angle at B . A rectangle is completed, $ABCD$. How does the



size of triangle I seem to compare with that of triangle II ? the base of the rectangle with the base of the triangle? the altitude?

EXERCISE 21

1. If the base BC were 12" and the altitude AB were 7", what would be the area of the rectangle? What would be the area of the triangle ABC ?

2. State a rule for finding the area of a right triangle.

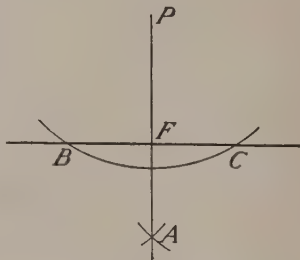
3. Express this rule as a formula.

Not all triangles are right triangles, and before we study the others we need to know Problem 5.

Construction Problem 5. *How to let fall a perpendicular from a point outside a line.*

Step 1. From P , the point from which we wish to let fall the perpendicular, an arc is drawn with any convenient radius, cutting the line at B and C .

Step 2. From B and C with equal radii draw arcs intersecting below the line.

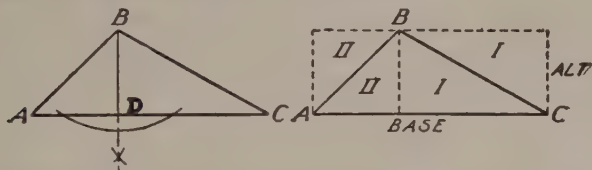


Step 3. Let the point of intersection A and join PA . Then PA is the required perpendicular.

PA forms with BC four right angles at F . The point F is called the foot of the perpendicular.

A carpenter when sawing a board through a given point gets a perpendicular by sliding his steel square along until the edge passes through the point. A draftsman uses right ("square-cornered") triangles in a similar way. In quick sketching you can imitate them by sliding a card or the corner of a sheet of paper along the line until it reaches the point through which you wish the perpendicular.

We can now divide any triangle into right triangles, by letting fall a perpendicular to the base.



What line is the altitude of the triangle ACB ? Explain how the figure was constructed. Into what two right triangles is the triangle ABC divided?

In the other diagram the two triangles marked I are equal and also the triangles marked II . What part of the large rectangle is included in the triangle ABC ?

EXERCISE 22

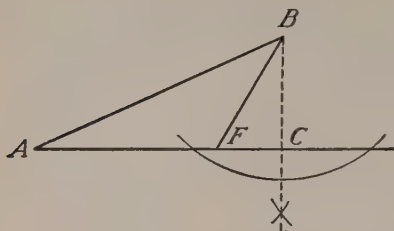
1. Give a formula for the area of any triangle in terms of its base and altitude.

2. Measure the base AC and the altitude BD of the triangle above, and calculate its area.

3. Draw any triangle having its longest side horizontal. With compasses let fall the altitude to the base. Measure the base and altitude and compute the area.

4. Construct an equilateral triangle with compasses, making each side 2" long. Let fall the altitude of the triangle. Take careful measurements and find the approximate area.

5. In some triangles we have to prolong the base in order to let fall the altitude. What line is the base of the triangle ABF in the figure? What line is the altitude? Measure them and find the area of the triangle ABF .



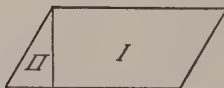
6. Construct an isosceles triangle with compasses, making the base $4''$ long and the arms each $2\frac{1}{2}''$ long. Let fall the altitude. Measure the altitude and find the area of the triangle.

7. Construct a triangle with base $1\frac{1}{2}''$ long and the other two sides $1''$ and $2''$. Prolong the base of the triangle, let fall the altitude, measure it, and calculate the area.

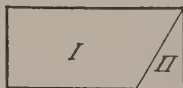
Find the area of the triangle in each example following:

	BASE	ALTITUDE	AREA
8.	$5''$	$3''$	
9.	$100'$	$50'$	
10.	$2.04''$	$3.25''$	
11.	$5\frac{1}{2}''$	$1\frac{1}{4}''$	
12.	12 meters	12 meters	

PARALLELOGRAMS



PARALLELOGRAM



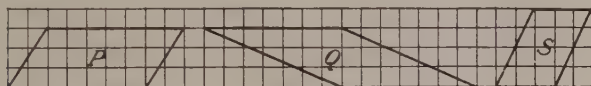
RECTANGLE

A parallelogram is a four-sided figure having its opposite sides parallel. After cutting off the triangle II at one end it may be placed at the other end to form a rectangle. It is

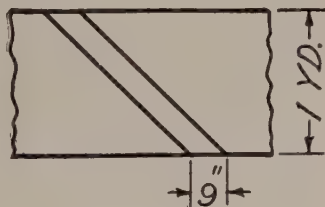
evident that the area of the parallelogram is the same as that of the rectangle, and is therefore equal to the product of its base and its altitude.

EXERCISE 23

1. State a formula for the area of a parallelogram (A, b, h).
2. If each ruled square represents a square inch, find the areas of P, Q , and S .

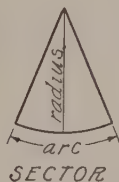
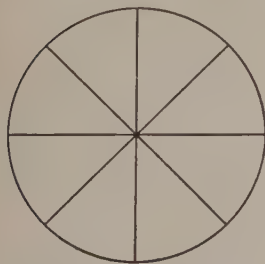


3. A ruffle is cut, as pictured, from a piece of cloth one yard wide. How many square feet of cloth are in the ruffle? how many square yards?



THE AREA OF A CIRCLE

The spokes of a wheel divide the circle into wedge-shaped figures called *sectors*.



A sector looks like a triangle with a curving base. The line in the figure corresponding to the altitude is equal to the radius, and it can be proved that the area

of a sector is equal to half the product of the length of its arc (base) and the radius (altitude).

EXERCISE 24¹

1. The length of the arc which forms the base of a sector is $6''$ and the radius of the sector is $10''$. What is the area?

2. Write a formula for the area of a sector (A) in terms of the length of its arc (l) and the radius R .

3. If $R = 14''$ find the circumference of the circle (use $\pi = 3\frac{1}{7}$), and the length of an arc of a sector if the arc is $\frac{1}{8}$ of the circumference.

4. A circle with radius $14''$ is divided into eight equal sectors. What is the area of each of the sectors?

5. Find the combined area of the eight sectors, *i.e.* of the entire circle.

6. Since the area of each sector is half the product of the radius by its arc, the sum of their areas is half the product of the radius by the sum of all the arcs, or the circumference. The area of a circle is therefore one-half the product of the radius by the circumference. Express this as a formula (A, R, C).

7. A circle with radius $7''$ is divided into 6 equal sectors:

(a) Find the circumference.

(b) How long is the arc of each sector?

(c) What is the area of each sector?

(d) What is the total area of the six sectors?

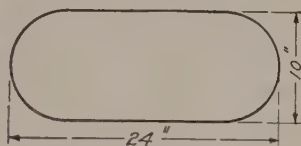
(e) Multiply the circumference by $\frac{1}{2}$ the radius and check with (d).

8. In the formula of Example 6 replace C by its value $2\pi R$ and show that the formula for the area of a circle in terms of its radius is $A = \pi R^2$.

For the circle in each of the examples 9–15 on the following page copy and fill in only the blanks that are indicated by a dash:

	R	D	CIRCUM- FERENCE	AREA	
9.	35"		220"	_____	Use $\pi = 3\frac{1}{7}$
10.	10"	_____	_____	_____	" "
11.	_____	12"	_____	_____	" "
12.	_____	$4\frac{1}{2}"$		_____	Use $\pi = 3.14$
13.	100'			_____	" "
14.	_____	3.34 cm.		_____	" "
15.	$\frac{D}{2}$	D	πD	_____	" "

16. The bottom of a wash boiler consists of a rectangle plus two semicircles. The width of the boiler is 10" and its total length 24". Find the area.



17. Find the perimeter of the bottom of the boiler.

18. Mr. Lee owns on a curving street a lot in the shape of a sector forming a quarter circle. How many square feet has he if the radius is 150'?

19. A window has the shape of a square, with a semicircular top added. If the window is 4' wide find its area in square feet.

*20 Find a formula for the area of the bottom of a wash boiler w inches wide, l inches in total length.

*21. Find also a formula for the perimeter of the bottom of the boiler (P , w , l).

*22. Find a formula for the area of a window like that in Example 19 if the window is w feet wide.

*23. Find a formula for the perimeter of the window.

*24. Find formulas for the perimeter and area of a window consisting of a rectangle (w' wide, h' high) plus a semicircular top.

CHAPTER III

NUMERICAL CALCULATIONS

SIGNIFICANT FIGURES

With ordinary measuring devices we can never measure a quantity exactly, and the numbers which we obtain by measurement are never more than approximations. After measuring with a steel tape the circumference of a tree, for example, it might be said that the circumference is 3.47'. We do not mean that the length is exactly 3.47', but that it is nearer 3.47' than 3.46' or 3.48'; it may be as small as 3.465' or as large as 3.474'. When we call it 3.47' we have what is termed a three-figure approximation; we have measured the length to three significant figures. The measuring of a tree is a problem of such a nature, however, that it was not really necessary to measure quite so accurately. We might call the girth of the tree 3.5'; in this case we would have two significant figures or a two-figure approximation.

If you spell $\frac{2}{3}$ of a test list correctly your mark is 67% (two significant figures).

If the New York Giants win $\frac{2}{3}$ of their baseball games this year their standing will be .667 (three significant figures).

Three pounds of crackers at $12\frac{1}{2}\text{¢}$ a pound will cost you 38 cents (two significant figures).

It is to be noticed that we choose for the final figure of any approximate number that figure which is the nearest to the true value. If the remaining fraction is a half or more it is the custom to use the next higher figure.

Example: Express $\frac{1}{6}$ as a decimal, correct to two significant figures.

Solution: $\frac{1}{6}$ means 1 divided by 6.

$$\begin{array}{r} .16\frac{2}{3} \\ 6 \overline{)1.00} \end{array} \quad \text{Ans. .17.}$$

Since $\frac{2}{3}$ is more than $\frac{1}{2}$, we write .17 because it is nearer the true value of $\frac{1}{6}$ than is .16.

The value of $\frac{1}{6}$ to three significant figures would be .167; to four figures it would be what?

Example: A meter is equal to 39.37 inches, correct to four significant figures. Write a three-figure approximation; a two-figure approximation.

To three figures, 39.4" Ans.

To two figures, 39" Ans.

EXERCISE 25

1. The value of π to seven figures is 3.141592. Write the value of π

- (a) correct to six significant figures;
- (b) correct to five significant figures;
- (c) correct to four significant figures;
- (d) correct to three significant figures.

2. On an article costing 60 cents the dealer made a profit of 25 cents. Find the per cent of gain, getting a two-figure approximation.

3. The Boston Red Sox won 14 games out of 30. Find the team's standing. (What fractional part is 14 of 30, to the nearest thousandth?)

4. Find to the nearest cent the cost of $2\frac{1}{4}$ yards of cloth at 49 cents a yard.

5. A survey of suburban land is supposed to be correct to five significant figures. If it were necessary to use π what value of π would be used?

6. Change the fractions $\frac{22}{7}$ and $\frac{355}{113}$ to decimal form. Compare the results with the value of π in Example 1 and tell to how many significant figures each fraction is a correct approximation of π .

7. Multiply 3.78 by 1.15 and leave only three significant figures in the answer.

8. Divide 50 by .785, giving an answer to two significant figures.

9. Calculate to three significant figures the value of C in the formula $C = 2\pi R$ if $R = 11.5''$.

Many numbers which do not result from measurements are of such a character that we are not interested in or concerned with the tiniest details, and we use only round numbers. When we speak of the circumference of the earth as 25,000 miles we are using a round number. We do not intend to say that it is precisely 25,000 miles, but that it is nearer 25,000 than it is 26,000 or 24,000. The three zeros are not significant but are used only to fill in the number of places necessary to indicate thousands. Only the 25, the first two figures, are significant. It is a two-figure approximation.

In each of the following statements only the 5 is significant, and each of the numbers is a one-figure approximation. The 0's serve merely to place the decimal point:

"The rose bush is 5 feet high."

"New York City contains 5,000,000 people."

"The thickness of this paper is .005."

A zero or cipher at the end may be significant in a decimal, and it is so in the following three statements:

"A meter is 39.370 inches."

"The cube root of 2 is 1.260."

" $1.48^2 = 2.190$."

A zero is always significant except when used to locate the

decimal point. For example the zeros underlined in the following are significant:

7.60, 103, 0.92, 35000, 50700.

In calculations involving measurements or any approximate number such as π , a most important principle is that *the accuracy of the result will not ordinarily be greater than that of the least accurate of the numbers used*. For instance, if the machinist with his finest tools had made a gear of diameter 3.222", he would not use a three-figure value of π such as $3\frac{1}{7}$ or 3.14 to find its circumference. On the other hand, if any measurement or approximation is roughly made, say to two figures only, it would be wasteful to express any other numbers in the same calculation correct to six or seven figures.

Example. The base and the altitude of a rectangle are found by measurement to be 2.2" and 1.6" respectively. Find the area.

Solution: Since the given numbers are correct to two significant figures only, their product will give the area of the rectangle correct to not more than two significant figures.

2.2

1.6

1 3 2

2 2

3.5 2

We are not sure of the third figure, the final 2, and the answer is therefore given to two figures only, 3.5 square inches.

Ans. 3.5 sq. inches.

EXERCISE 26

1. A practical farmer would not calculate the capacity of a bin in bushels by allowing 2150.42 cubic inches for each bushel. What round number might he use?

2. In finding the circumferences of hundreds of forest trees an inexperienced forester multiplied the diameter by 3.1416. What value of π would you recommend that he use?

3. In calculating the area of some land not far from Broadway, a New York City draftsman used $3\frac{1}{7}$ for π . What value should he have used?

4. The weight of a cubic inch of cast iron, correct to two significant figures, is 0.26 lb. Find the weight of an iron bar containing 42 cubic inches. Answer to two significant figures.

5. Find the weight of a cubic foot of cast iron. Answer in round numbers, to two significant figures only.

6. The base of a triangle is 19" and the altitude is 17.3" by measurement. Find the area of the triangle.

EVALUATION OF FORMULAS INVOLVING SQUARE AND CUBE ROOTS

The square root of any number is that number which, when multiplied by itself, produces the given number. Thus the square root of 36 is 6, since 6 times 6 gives 36. In the abbreviated form the statement is written $\sqrt{36} = 6$.

EXERCISE 27

1. What number multiplied by itself gives 100? What then is the square root of 100?

2. What is the square root of 81? of 144?

3. Show by multiplication that 8.5 is the square root of 72.25.

The cube root of a number is that number which, when cubed, will produce the given number. The cube root of 125 is 5 because 5^3 , that is $5 \times 5 \times 5$, is 125. $\sqrt[3]{125} = 5$.

4. What is the cube root of 8? of 216?

5. By multiplication prove or disprove that 52 is the cube root of 140,608.

When $a = 4$, $b = 8$, $c = 9$, find the value of:

- | | | |
|---------------------|---------------------------|-----------------------|
| 6. $\sqrt{a}$ | 12. $\sqrt{4a + c}$ | 18. $\sqrt{a^3}$ |
| 7. $\sqrt{ac}$ | 13. $\sqrt{4a + c}$ | 19. $\sqrt[3]{b}$ |
| 8. $\sqrt{2b}$ | 14. $\sqrt{\frac{1}{a}}$ | 20. $\sqrt[3]{3c}$ |
| 9. $\sqrt{4c}$ | 15. $\frac{b}{\sqrt{4c}}$ | 21. $b\sqrt[3]{2a}$ |
| 10. $\sqrt{a^2c}$ | 16. $\sqrt{9a^2 + c^2}$ | 22. $a\sqrt[3]{b^2}$ |
| 11. $4a + \sqrt{c}$ | 17. $\sqrt{100c}$ | 23. $\sqrt[3]{1000b}$ |

The Use of Tables—Spheres. On page 377 will be found a table giving the square, the cube, the square root, and the cube root of each integer from 1 to 50. The table is printed in order to save you a great deal of time and labor in making calculations, as well as to help you to avoid mistakes. The square roots and the cube roots are given correct to four significant figures.

EXERCISE 28

1. Find the square of 48 by multiplication. Check by finding it in the table.

2. From the table obtain the square root of 37, the square root of 2, and the square root of 45. Add the three square roots.

3. Write down from the table the square of 43 and the square of 34, and find the difference of the two squares.

4. Find the value of $\sqrt{29}$ from the table and verify it by multiplying this value by itself.

5. Find the value of $\sqrt[3]{29}$ from the table and verify it by cubing the value found, by multiplication.

6. What is the sum of the squares of 28 and 49?

7. What is the difference of the cubes of 35 and 28?

8. If $a = 33$ and $b = 28$, find the value of $a^3 + b^2$.

9. If $a = 33$ and $b = 28$, find the value of $\sqrt[3]{a} + \sqrt{b}$.

Since the area of a square is found by squaring the length of a side of the square, if we know the area of a square we can find the side by taking the square root. If the area of a square is 49 sq. inches, the side of the square is $\sqrt{49}$ or 7 in.

10. Find the side of a square whose area is 400 sq. inches.
11. What is the side of a square field, area 40 sq. rods?
12. Write a formula for the side (s) of a square in terms of the area (A).

13. To find the volume of a cube, we cubed the length of its edge. If the edge of a cube is 3 inches its volume is 3^3 or 27 cubic inches. What is the edge of a cube if the volume is 64 cubic inches? 1728 cubic inches?

14. If a cube contains exactly 2 cubic inches, find the edge of the cube from the table.

15. Write a formula for the edge (e) of a cube in terms of its volume (V).

16. Using the table, find the value of A in the formula $A = \pi R^2$ if $R = 46$, $\pi = 3.142$.

17. Using the table, find c^2 in the formula $c^2 = a^2 + b^2$ if $a = 41$ and $b = 37$.

18. Using the table, see if you can obtain the value of t from the formula $t = \sqrt{\frac{2s}{g}}$ if $s = 992$ and $g = 32$.

19. Archimedes, one of the ablest mathematicians of ancient times, was delighted when he discovered that the area of the surface of a sphere is exactly four times that of a circle having the same radius. Write a formula for the surface (S) of a sphere in terms of its radius.



A HEMISPHERE

20. Copy and complete: The curved surface of a hemisphere is ——— that of the sphere, and is equal to ——— R^2 . The curved surface of a hemisphere is therefore exactly ——— as large as its flat surface.

21. Replace the radius in Example 19 by its value in terms of the diameter and show that the surface of a sphere is π times the square of its diameter. Write a formula for S in terms of D .

22. Using this formula evaluate S if $D = 42''$ and $\pi = 3\frac{1}{7}$.

23. Archimedes also discovered that the volume of a sphere is one-third the product of its surface times its radius. Using the surface formula in Example 19, try to derive a formula for V in terms of R .

24. By substituting $R = \frac{D}{2}$ in the preceding formula for V , show that $V = \frac{1}{6} \pi D^3$.

25. Change $\frac{1}{6} \pi$ to a decimal (four significant figures) and substitute this decimal in the formula of the preceding example.

26. With the help of the tables find the volume of a sphere of diameter $35''$. (Use $\pi = 3\frac{1}{7}$.)

27. Find the volume of a sphere in cubic inches if the diameter is one foot.

28. Find the volume in cubic inches if the radius is one foot, and divide the result by the result in Example 27.

29. An old cannon ball was $16''$ in diameter. What did it weigh if iron weighs $\frac{1}{4}$ lb. per cubic inch?

30. A ball bearing measures $.045$ in. in diameter. Find its surface to three significant figures.

31. Find the volume of the bearing to three significant figures.

32. A cast-iron sphere is $5.5''$ in diameter. What does it weigh if cast iron weighs 0.26 lb. per cubic inch?

33. What is the weight of the sheet lead which must be used to cover a dome in the shape of a hemisphere $48'$ in diameter, if sheet lead weighs 5 lb. per square foot?

34. Basing your work on the tables as given, complete the shorter table below, giving the square roots and cube

roots of the numbers from 1 to 10 to three significant figures:

n	$\sqrt{n}$	$\sqrt[3]{n}$
NUMBER	SQUARE ROOT	CUBE ROOT
1	1.00	1.00
2	1.41	1.26
3	1.73	etc.

35. In rough work two significant figures often give sufficiently close approximations. Write two-figure square roots of the numbers from 1 to 10.

*36. The tables of cubes on page 377 may be used to approximate the cube roots of many numbers larger than 50. To find the cube root of 80,000 locate in the n^3 column the number nearest to 80,000 (you will find 79,507). The cube root of 80,000 is approximately 43. Approximate to two figures the following:

(a) $\sqrt[3]{110,000}$

(b) $\sqrt[3]{39,500}$

(c) $\sqrt[3]{13,700}$

(d) A cubical box contains one bushel, 2150 cubic inches. Find the approximate depth of the box.

EQUATIONS

The formulas and many of the algebraic statements of the preceding pages are in the form of equations, such as $S = 4\pi R^2$, $p = 2l + 2w$, $x + 5 = 20$, etc. The study of equations is most important, and learning how to solve equations in order to find the answers to problems is one of the chief purposes of your study of algebra. In this chapter

we will take up only the simplest kind of equations, gradually extending our knowledge of them as we proceed through the book.

If it is known that four times the side of a square is equal to 36, we may write:

$$4s = 36$$

If, however, we know that four times a certain number is 36, we can easily find the number, and therefore we write $s = 9$. How did we get the 9?

$$\begin{array}{l} \text{Try these: } 4s = 144 \\ s = ? \end{array}$$

$$\begin{array}{l} 4s = 48 \\ s = ? \end{array}$$

$$\begin{array}{l} 4s = 35 \\ s = ? \end{array}$$

In each case what is the coefficient of s ? The result is obtained by dividing each side of the equation by the coefficient, 4.

EXERCISE 29

1. Divide each side of the equation $12x = 60$ by 12.
2. Divide each side of the equation $9r = 100$ by the coefficient of r .

This finding the value of a letter in an equation is called *solving the equation*. Solve the following equations by dividing each side of the equation by the coefficient of the letter:

$$3. \quad 5x = 100$$

$$14. \quad 15p = .90$$

$$4. \quad 4s = 48$$

$$15. \quad 6n = 1200$$

$$5. \quad 13x = 26$$

$$16. \quad 3.5t = 70$$

$$6. \quad 3c = 75$$

$$17. \quad 1.7p = 17$$

$$7. \quad 7x = 101$$

$$18. \quad \frac{1}{3}n = 10$$

$$8. \quad 12f = 36$$

$$19. \quad \frac{3}{4}x = 24$$

$$9. \quad 2y = 7800$$

$$20. \quad 1.06p = 530$$

$$10. \quad 8F = 1760$$

$$21. \quad 4x = 3$$

$$11. \quad 6m = .042$$

$$22. \quad 25x = 1$$

$$12. \quad 3x = 10$$

$$23. \quad 3\frac{1}{7}D = 88''$$

$$13. \quad 7n = 100$$

$$24. \quad 3.14D = 78.5'$$

25. $5y = 2$ (as a decimal).
26. $6r = 5$ (to the nearest thousandth).
27. $6x = 10$ (to three significant figures).
28. $1.6x = 5$ (to two significant figures).
29. $7w = 1000$ (to four significant figures).
30. $5.8l = 144$ (to three significant figures).
31. Write a formula for the length of a rectangle, given A and w . Find l if $w = 1.5$ and $A = 27$.
32. The solution of equations like Examples 23 and 24 gives the diameter of a circle when the circumference is known. State a rule for finding the diameter. Express the rule as a formula.
33. Find the diameter if $C = 154$. Use $\pi = \frac{22}{7}$.
34. Find the diameter, correct to three significant figures, if $C = 36''$.
35. Find the radius, correct to four significant figures, if the circumference of a circle measures $100'$.
36. What is the diameter of a tree if the circumference is $55''$?
37. Find the diameter of the earth at the equator if the equator measures 25,000 miles.
38. The circumference of a piston is 226.19 inches. Find the diameter to four significant figures.
39. Find the length of a rectangle if the width is 22.5 inches and the area 311 square inches.
40. $0.7854x = 25$. Find the value of x to four significant figures.
41. $2.173y = .3786$. Find y to four significant figures.

When several terms involving the unknown quantity appear in the left side or left member of an equation, it is necessary first to combine these into a single term; then to

divide both left and right members of the equation by the coefficient of the unknown quantity as before.

To check the solution of an equation it is necessary to substitute the value obtained in place of the unknown in the given equation, and to show that the resulting values of the two members are equal.

Example 1: Solve for x : $3x + 2x = 70$

Solution:

$$\begin{array}{rcl} & 3x + 2x = 70 & \\ \text{Combining the terms in } x, & 5x = 70 & \\ \text{Dividing both the } 5x \text{ and} & & \\ \text{the } 70 \text{ by } 5, & x = 14 & \end{array}$$

Check:

Left member:		Right member:
$3x + 2x$		70
$3(14) + 2(14)$		70
$42 + 28$		70
70	$\longleftrightarrow$	70

Example 2: Solve for n : $n + 6n + 5n = 100$

Solution:

$$\begin{array}{rcl} & n + 6n + 5n = 100 & \\ \text{Combining the terms in } n, & 12n = 100 & \\ \text{Dividing each member by } 12, & n = 8\frac{1}{3} & \end{array}$$

Check:

Left member:		Right member:
$n + 6n + 5n$		100
$8\frac{1}{3} + 6(8\frac{1}{3}) + 5(8\frac{1}{3})$		100
$8\frac{1}{3} + 50 + 41\frac{2}{3}$		100
$50 + 50$		100
100	$\longleftrightarrow$	100

Example 3: Solve to the nearest cent: $c + 35\%$ of $c = \$200$.

Solution:

$$c + .35c = 200$$

$$1.35c = 200$$

$$c = 200 \div 1.35$$

$$= 148.15$$

$$\therefore c = \$148.15 \quad \text{Ans.}$$

$$\begin{array}{r}
 148.14\overline{1\frac{10}{35}} \\
 1.35 \overline{) 200.00} \\
 \underline{135} \\
 650 \\
 \underline{540} \\
 1100 \\
 \underline{1080} \\
 200 \\
 \underline{135} \\
 650 \\
 \underline{540} \\
 110
 \end{array}$$

Check:

Left	Right
member:	member:
$c + .35c$	$\$200 \quad \148.15
$\$148.15 + (.35) (\$148.15)$	$200 \quad .35$
$148.15 + 51.85$	$200 \quad \underline{74075}$
	44445
$200 \longleftrightarrow 200$	$\$51.8525 \quad \148.15
	or 51.85
	$\$51.85 \quad \200.00

EXERCISE 30

Solve and check:

1. $2n + 3n = 170$

2. $7x - 5x = 18$

3. $100y + 50y = 600$

4. $15c + 8c - 10c = 52$

5. $x + 2x + 3x = 30$

6. $p + .04p = 260$

7. $c + 50\%$ of $c = 48$

8. $4w + 2w = 240$

9. $\frac{1}{2}r - \frac{1}{3}r = 10$ 11. $x + \frac{1}{2}x + \frac{1}{4}x = 35$
 10. $.04p + .035p = 375$ 12. $x - \frac{1}{6}x = 57$
 13. $b + .30b = \$.60$ (to nearest cent). Ans. \$.46
 14. $2D + \pi D = 1760$ (to three significant figures)
 15. $\frac{1}{2}\pi D + 3D = 100$ (to three significant figures)

Problems Solved by Equations

Many problems are solved more conveniently by means of equations than by the method of arithmetic. Many other problems which could not be solved at all by arithmetic alone are easily worked by algebra. The difficulty for beginners is likely to be in planning the proper equation,—i.e. in translating from the words of the statement into the symbols of algebra. The work already done in Chapter I with formulas will help now.

In each problem represent the unknown number by its initial letter, or by x or n . Follow the general plan of the model examples and arrange neatly the work of your solution. To check a problem make sure that your numerical answer, when put into the original statement in place of the words, makes that statement true.

Example 1: What number added to three times itself will make the sum 24?

Solution: Let x = the number.

What number added to three times itself will make the

$$x \quad + \quad 3 \quad \cdot \quad x \quad =$$

sum 24?

24

$$x + 3x = 24$$

$$4x = 24$$

$$x = 6 \quad \text{Ans.}$$

Check: 6 added to 3 times 6, or
 6 added to 18, makes 24.

Example 2: Find a number whose double exceeds half the number by 18.

Solution: Let n = the number.

Find a number whose double exceeds half the number by 18.

$$2n \quad - \quad \frac{1}{2}n \quad = 18$$

$$1\frac{1}{2}n = 18$$

$$n = 18 \div 1\frac{1}{2}$$

$$n = 12 \quad \text{Ans.}$$

Check: The double of 12 exceeds $\frac{1}{2}$ of 12, or 24 exceeds 6, by 18.

EXERCISE 31

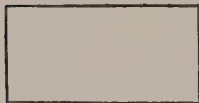
Solve each problem by means of an equation:

1. If x represents a number, what equation says that five times the number equals 40? Solve the equation.

2. If s represents the side of a square, what equation states that four times the side equals 24? Solve the equation.

3. If to a certain number we add twice the number the sum will be 18. (Let n = the number, write the equation and solve.)

4. If from seven times a number we subtract three times the number the result will be 12. Find the number.



5. Find a number whose double exceeds the number itself by 15.

6. A rectangle is twice as long as it is wide. Find the width if the sum of the four sides is 60 inches.

7. A rectangle is 3 times as long as it is wide. Find the width if the perimeter is 32 centimeters.

8. How can a man divide \$96 between his wife and his son so that the wife receives 5 times as much as the son?

9. I am thinking of a number which when subtracted from four times itself gives 48. What is the number?

10. James bought two pencils for 12 cents, paying five times as much for one as for the other. What did the cheaper pencil cost?

11. For luncheon May spent 20 cents, buying a plate of ice cream and a sandwich. If the ice cream cost three times as much as the sandwich, find the cost of the latter.

12. An expert advises that a house should properly be worth three times as much as the lot. If this is true of a house and lot costing together \$9000, what was the cost of the lot?

13. Three boys sold 90 tickets to a football game. If John sold twice as many as Bob, and Dick sold three times as many as Bob, how many did Bob sell?

14. A hospital fund of \$40,000 is to be raised by three towns. The largest town agrees to give five times as much as the smallest, and the third town twice as much as the smallest. How much must each town give?

15. Divide 35 cents among three girls in such a way as to give Alice twice as much as Helen, and Mary twice as much as Alice.

16. Each of the two equal sides of an isosceles triangle is twice as long as the base. Find the base if the perimeter is 35 inches.

17. The perimeter of an isosceles triangle ABC is 60 inches. Find the three sides if BC is five times as long as AB , and $AC = BC$.



18. A bicycle was sold for \$42, the selling price being four-thirds of the cost. Find the cost.

19. A house was sold for \$6000, which was one-fifth less than it cost. Find the cost.

20. An automobile was sold for \$550 at a gain of 25 per cent. What was the cost of the automobile?

21. If the population of Fordville in 1920 was 4800, representing an increase of 50% over its population in 1910, what was the population in 1910?

22. In the formula $lwh = V$, find h if $V = 1728$, $l = 4$, and $h = 2$.

23. A board, one inch or more in thickness, which contains 144 cubic inches of lumber, is equal to one board foot of lumber. Find l if $w = 6''$, $h = 1''$, $V = 144$ cubic inches.

24. Find the length of a board containing one board foot if the board is $4''$ wide and $1''$ thick.

*25. Find the dimensions of several different boards from 3 to 16 inches wide each to contain 1 board foot, and each to be $1''$ thick. Make a little sketch of each board using $\frac{1}{8}''$ on the sketch to represent $1''$ on the board.

*26. A box $10''$ square is to hold a bushel, 2150.4 cubic inches. How deep must the box be made?

*27. A standard bushel box is $24''$ long and $11\frac{1}{5}''$ wide. How deep is it?

*28. Another standard bushel is $8''$ deep and $16.8''$ long. How wide is it?

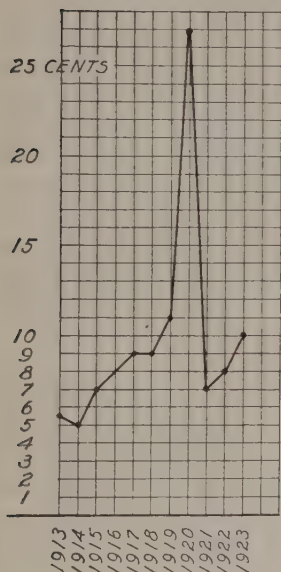
*29. A box $8''$ square contains $\frac{1}{2}$ peck or $\frac{1}{8}$ bushel. How deep is it?

*30. A box $4''$ square is to hold a quart or $\frac{1}{8}$ peck. How deep shall it be made?

*31. If a ton of a certain kind of coal takes up 45 cubic feet, state a formula for the number of tons (T) which a coal bin l' long, w' wide h' high, will contain.

CHAPTER IV

GRAPHS AND NEGATIVE NUMBERS



In this graph from left to right the horizontal lines are marked off in spaces, one space for each year from 1913 to 1923. The vertical lines are marked off in spaces to represent the number of cents in the price of a pound of sugar.

In what year did sugar sell at the highest price? What was the highest price?

What was the lowest price at which it sold? In what year was this?

What was the price of sugar in 1917?

In what year did sugar sell for 11 cents per pound?

What was the total increase in the price of sugar from 1914 to 1920? The total decrease from 1920 to 1921?

EXERCISE 32

1. Number the spaces on a sheet of cross section paper as they are numbered in the sugar diagram and draw a graph

showing the price of a quart of milk for the ten years 1914-1923.

Year.....	1914	'15	'16	'17	'18	'19	'20	'21	'22	'23
Price of Milk..	9	9	9	11	14	15	17	15	13	14

2. Draw a graph showing the New York City death rate per 1000 inhabitants. Mark off the given dates along the bottom of your paper, $\frac{1}{2}$ " apart.

Year.....	1904	'06	'08	'10	'12	'14	'16	'18	'20	'22
N.Y. Death Rate	20	18	16	16	15	15	15	18	13	11

After drawing the graph answer (a) and (b).

(a) Doctors and health officials are proud of the recent history of the death rate. Can you tell why?

(b) Does the graph show in what recent year there was a serious epidemic of influenza?

3. Draw a graph of the New York City death rate per 100,000 inhabitants from street accidents, and ask yourself whether this also is one of which to be proud.

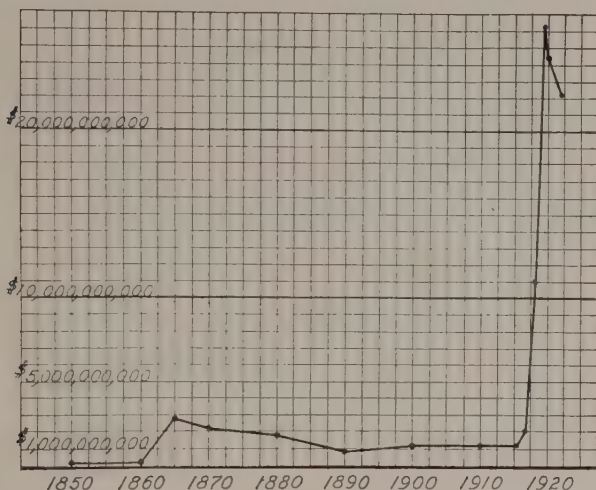
N. Y. Death Rate from Street Accidents in

Year.....	1904	'06	'08	'10	'12	'14	'16	'18	'20	'22
Rate.....	9	13	11	9	10	12	12	17	16	17

4. Draw a graph of John's monthly marks in mathematics last year: 7, 6, 5, 7, 7, 8, 8, 9, 7, 8 (on a scale of 10). If possible use your own marks in place of John's. By using different colors you could represent all your subjects of study on the graph.

When drawing graphs of large numbers we have to choose a suitable scale so that all the numbers may be represented within the limits of our drawing. The scale will vary; it may be 1 centimeter to represent 20 pounds, 1 inch to represent 1000 people, or 1 space to represent 1 billion dollars, as

in this graph of the U. S. Debt. The choice of too small a scale must be avoided, for it might flatten out a graph so that significant changes could not be clearly perceived.



In what year did our debt reach its highest point? How many billion dollars was the maximum?

From the graph what fraction of our debt do you suppose was the result of wars—a half? three-quarters? nine-tenths?

Much of the debt is in the form of Liberty Bonds. How many fifty-dollar Liberty Bonds would amount to twenty billion dollars?

What horizontal scale might be used to represent the U. S. Debt for 130 years, from 1790 to 1920, if your paper is 7" wide?

What vertical scale might be used to draw a graph of the enrollment from year to year of a high school which now has 956 pupils, if your paper is 10" high? if your paper is 5" high?

EXERCISE 33

1. The anthracite coal production for every second year since 1900 is given here in millions of tons. Draw a graph using as the vertical scale 1 inch for each 20 million tons.

Year.....	1900	'02	'04	'06	'08	'10	'12	'14	'16	'18	'20	'22
Anthracite Produced }	51	37	65	64	74	75	75	81	78	88	80	49

(a) How does your graph show you in what years there were great coal strikes?

(b) In what year was the largest amount produced?

2. The average temperatures in Chicago for each month of the year beginning with January are 24° , 25° , 34° , 46° , 56° , 66° , 72° , 71° , 65° , 53° , 39° , 29° . Draw a graph letting 1" vertically represent 20° , and $\frac{1}{2}$ " horizontally represent 1 month.

3. As the price of an article goes up the demand for it is likely to fall off. A school lunchroom had been selling sandwiches for 5 cents each, at a small loss; the management decided to raise the price 1 cent each week up to 10 cents. Draw a graph to show the results, choosing a suitable vertical scale to represent the sales, and marking off the prices $\frac{1}{2}$ " apart horizontally:

Week of.....	Oct. 1	Oct. 8	Oct. 15	Oct. 22	Oct. 29	Nov. 5
Price.....	5	6	7	8	9	10
Number Sold.	2000	1700	1500	1300	800	500

(a) Calculate the amount received for sandwiches each week.

(b) What price would you advise the management to adopt? Why?

A vivid comparison of two quantities may be obtained by representing both of them on the same graph.

4. In one diagram draw graphs to represent the average monthly rainfall in San Francisco and in Colon. Vertical

scale: let 1" represent 5 inches of rainfall. Use a dotted line or a different color for one of the graphs.

	Jan.	Feb.	Mar.	April	May	June
San Francisco.....	4.3	3.8	3.1	1.8	0.8	0.2
Colon.....	4.0	1.5	1.6	4.3	12	13
	July	Aug.	Sept.	Oct.	Nov.	Dec.
San Francisco.....	0.0	0.0	0.3	1.3	2.5	4.2
Colon.....	17	15	13	14	20	12

5. Draw in the same diagram graphs showing the monthly receipts and expenses of a hotel in a seaside city. Vertical scale 1" to \$1000.

MONTH	RECEIPTS	EXPENSES
January.....	\$1100	\$1500
February.....	900	1400
March.....	1000	1500
April.....	1400	1600
May.....	1500	1500
June.....	3000	2000
July.....	6000	4000
August.....	7500	5000
September.....	3500	3000
October.....	2000	1800
November.....	1300	1600
December.....	1500	1600

After drawing the graph, answer these questions:

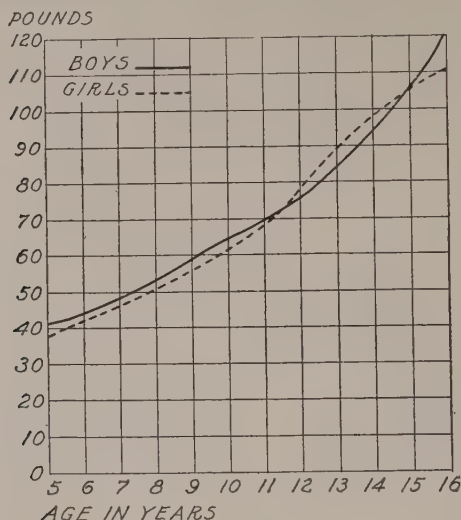
(a) In which month were the receipts greatest? Least?

(b) In which month is the greatest margin between receipts and expenses?

(c) In which months does the graph of receipts run below that of expenses?

(d) At which months do the graphs cross? What does this indicate as to receipts and expenditures?

COMPARISON OF WEIGHTS OF BOYS AND GIRLS



This graph is not a broken or jagged line—as have been those preceding, the lines being curved because of the fact that the weights would not change suddenly, but would increase gradually from one year to the next. On the vertical lines each unit or space represents 5 pounds, and this is the vertical scale; the use of 1 space for each pound would of course make the graph inconveniently large.

What should a boy weigh at age 11? at age 13?

A 15-year-old girl weighs 115 pounds. How much is she above the standard weight or below the standard weight? What about your own weight?

At what age do girls first catch up with boys in weight? At what age do the boys again catch up with the girls?

After the scale is chosen and points are located on such a graph, draw a curved line through the points as smoothly as you can. Sometimes a flexible piece of metal can be used

by bending it so as to pass through three or more successive points. Rooms for mechanical drawing are equipped with wooden or celluloid curves.

EXERCISE 34

1. The population of the United States for each tenth year beginning with 1790 has been, in millions, 4, 5, 7, 10, 13, 17, 23, 31, 39, 50, 63, 76, 92, 106. Use a vertical scale of 1" for 10 million or for 20 million, and draw the graph.

2. Draw a graph showing standard heights, using a scale similar to that of the weight graph.

Ages..	5	6	7	8	9	10	11	12	13	14	15	16
Boys..	42	44	46	48	50	53	55	57	60	62	65	67
Girls..	41	43	45	48	50	53	56	58	60	61	62	64

3. In one diagram draw graphs representing the growth of population of the two states. (Vertical scale 1" for 1,000,000.)

POPULATION OF SOUTH CAROLINA		POPULATION OF ILLINOIS
1830	580 thousand	150 thousand
1840	590	500
1850	670	850
1860	704	1700
1870	706	2500
1880	1000	3100
1890	1150	3800
1900	1350	4800
1910	1500	5600
1920	1700	6500

4. Draw graphs showing the comparative growth in population of Detroit and Boston (000's omitted).

	1860	1870	1880	1890	1900	1910	1920
Detroit.....	45	80	120	200	290	470	990
Boston.....	180	250	360	450	560	670	750

5. The average cost of all articles of food combined, taking the average in 1913 as 100, is given in the table. Mark the bottom line of your graph 80, use a scale of 1" for 50 points, and draw a graph.

1907	1908	1909	1910	1911	1912	1913	1914	1915
82	84	89	93	92	98	100	102	101
1916	1917	1918	1919	1920	1921	1922	1923	
114	146	168	186	203	153	142	147	

6. The weekly wages paid to factory workers in N. Y. State (over a million of them) have averaged as below. After drawing the graph compare with Example 5 and consider what years have been best and worst for the workers.

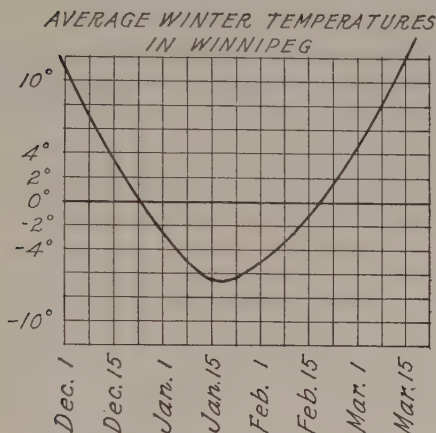
1914	1915	1916	1917	1918	1919	1920	1921	1922	1923
\$12.50	12.80	14.40	16.40	20.40	23.50	28.20	25.70	25.00	26.00

7. If possible look up in the U. S. Census reports or in the N. Y. World Almanac the population of your own state, city, or town since 1850 and draw a graph.

*8. From the Statesmen's Year Book or an encyclopedia obtain the population of France and of Germany since 1830, and perhaps of other countries, including our own, and draw comparative graphs in different colors on a large sheet of paper.

*9. Draw a graph showing the high school enrollment in your city or town for the last 10 years.

NEGATIVE NUMBERS



What is the average temperature in Winnipeg on March 1st?

At what time in the early winter is the temperature about the same as on March 1st?

Which is the coldest month of the year?

What is the average temperature for that month?

What is the average temperature on January 1st?

Instead of writing the January 1st temperature as "2 below zero" it is customary in weather reports to write it as -2 . The $-$ sign is necessary to distinguish it from 2 above (which may be written $+2^\circ$) and a number so written is called a negative number. It is read "minus 2." Any temperature below zero is usually printed as a negative number, with a minus sign.

EXERCISE 35

1. Write as a negative number the average Winnipeg temperature, (a) on January 15th; (b) on February 1st.

2. "Water freezes at $+32^{\circ}$, mercury at -40° ." Rewrite this sentence, using the words *above zero* or *below zero* in place of the $+$ and $-$ signs.

3. "In Boston on the coldest day of the winter the temperature went down to 14 below zero; in Montpelier on the same day it went to 40 below zero." Write these temperatures with $+$ or $-$ signs.

4. Draw a graph to show the temperature changes on a certain January day.

12	2 A.M.	4	6	8	10	12	2 P.M.	4	6	8	10	12
4	3	1	0	2	6	11	9	4	-1	-4	-7	-9

5. Draw a graph based on the following average temperatures in Winnipeg for all the months of the year beginning with January: -7° , -1° , $+12^{\circ}$, 36° , 51° , 62° , 66° , 63° , 52° , 39° , 18° , 4° .

6. The range of temperature means the difference between the highest temperature given and the lowest. In Example 4 the highest given is 11° and the lowest -9° ; what was the range for the day?

7. What is the range of the average temperatures in Winnipeg in Example 5?

8. The examination marks for a class ranged from 35 to 92. What was the total range?

9. "My business last week ranged from a loss of \$5 on Tuesday to a gain of \$30 on Saturday." How many dollars better was Saturday's business than Tuesday's?

10. If the temperature today is 9° what will it be tomorrow, (a) if it is 15° warmer? (b) if it is 15° colder?

11. If the temperature yesterday was 5° what is it today, (a) if it has risen 16° ? (b) if it has fallen 5° ? (c) if it has fallen 9° ?

12. (a) Add 16 and 5. (b) Subtract 5 from 5. (c) Subtract 9 from 5.

You have just been asked to subtract the larger of two numbers (9) from the smaller (5). In ordinary arithmetic this would be impossible, but in algebra negative numbers are introduced and such subtractions are frequently performed. To help you to understand such examples you are advised to refer each one to a scale of numbers like the thermometer scale. In order to subtract 9 from 5 find a number which is 9 spaces down from 5, the answer being -4 .

When subtracting a positive number, count down on the scale:

Example: Subtract 3 from 7.

Starting at 7 count down 3 spaces. The answer is 4.

Example: Subtract 5 from -2 .

Starting at -2 count down 5 spaces. The answer is -7 .

Example: From 0 subtract 3.

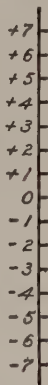
From 0 count down 3 spaces. The answer is -3 .

When adding a positive number count up on the scale.

Example: Add 7 to -4 .

Starting at -4 count up 7 spaces. The answer is 3.

After a few examples you will be able to work rapidly, without actually counting each of the spaces.



EXERCISE 36

1. Draw a number scale like the one here, using $\frac{1}{4}$ " spaces.
2. If the temperature is now -5° , what will it be after a rise of 3° ? of 5° ? of 12° ?
3. Add 12 to -5 . Add $+5$ to -5 .
4. Mr. Manning was notified by his bank that his account was overdrawn \$15. He at once made a deposit of \$100. What balance did that give him?
5. Add 100 to -15 . Add $+10$ to -15 .

6. Mrs. Richman has a balance of \$27.50. She makes out a check for \$40. How much will this overdraw her account?

7. Subtract 4° from 32° . Subtract 40 from 40.

8. In New Haven the temperature on a certain day was 6° above zero. In Bangor it was 20° colder. What was the temperature in Bangor?

9. Subtract 20 from 6. Subtract 6 from 20.

10. One day in 1923 the thermometer in New Orleans stood at 42° . In Des Moines it was exactly 51° colder. What was the temperature in Des Moines?

11. From 42 subtract 53. What number is 15 less than -5 ?

12. What is $42 - 50$? $5 - 15$?

13. Mr. Clark failed in business in 1922, owing \$20,000 which he could not pay. In 1923 he started out again and succeeded in making \$45,000. He paid the 1922 debts and had how much remaining?

14. Add 45 and -20 . From 45 subtract 20.

(Adding -20 gives the same result as subtracting 20. When adding a negative number the example is the same as when subtracting a positive number; count down on the number scale.)

15. A loss of \$5 on Tuesday and a loss of \$3 on Wednesday would make what total loss for the two days?

16. Add -5 and -3 . Add -4 , -1 , and -8 .

17. In keeping score for a certain game each losing player forfeits a number of points, the winner adding to his score an amount equal to the combined forfeits of the others. Thus if A wins the first round while B forfeits 20 and C 15, the scores will be

A	B	C
35	-20	-15

What will be the score after the next round if A forfeits 16 and B forfeits 20, the winner being C ?

18. Find the scores after each round of this game:

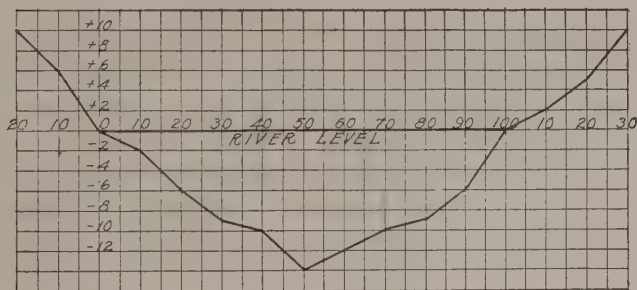
- (a) First round—*A* loses 10, *B* loses 15, *C* wins.
- (b) Second round—*A* loses 22, *B* wins, *C* loses 18.
- (c) Third round—*A* wins, *B* loses 5, *C* loses 13.
- (d) Last round—*A* wins, *B* loses 12, *C* loses 8.

Who wins the game?

19. Add 35, -20 , and -15 . (It is best to combine the two minus numbers first, since numbers may be added in any order.)

20. If you add the scores of all three players after any round, what is the sum?

Engineers interested in flood prevention made this profile section across a stream 100' wide:



What is the horizontal scale of the profile? the vertical scale?

Why do you suppose the vertical scale is chosen so much larger than the horizontal?

What is the greatest depth of the stream? How far is it out from the bank?

EXERCISE 37

1. (a) How wide would the stream shown in the graph be if in time of flood the river level rose 10 feet?

(b) The channel is that part of the river which is at least 10 feet deep. How wide is the channel?

(c) What is the difference in elevation between a point on the river bed 10' from the left bank and a point 10' from the right bank?

(d) What is the difference in elevation between a point 20' back from the shore on the left bank and a point 20' out from the shore on the river bed?

2. At another place along the river elevations at intervals of 10' (starting 30' back from the bank) were 10, 6, 2, 0, -3, -6, -11, -14, -10, -6, -2, 0, 10 ft. Draw a profile section.

(a) How wide is the river at this place? the channel? Answer approximately.

3. Find the difference in elevation or altitude between the summit of Pike's Peak, 19,108' above sea level, and that of Mt. Everest, 29,000' above sea level.

4. Find the difference in elevation between each of the two mountains named and the level of the Dead Sea in Palestine, 1292' below sea level.

5. New York City water comes mostly from the Ashokan Reservoir in the Catskill Mountains, 590' above sea level; it flows under the Hudson River in a tunnel 1114' below sea level, thence to the Kensico Reservoir 355' above sea level. The terminal reservoir, Silver Lake, on Staten Island, is 228' above sea level:

(a) How many feet does the water fall between Ashokan and the tunnel?

(b) How many feet does it rise from the tunnel to Kensico?

(c) What is the difference in elevation between Ashokan and Kensico? between the tunnel and Silver Lake?

6. Find the range of temperature in each of the places named:

PLACE	MAXIMUM	MINIMUM	RANGE
Omaha.....	110°	-32°	142°
San Francisco.....	101°	29°	
Block Island.....	92°	-6°	
Seattle.....	96°	11°	
Philadelphia.....	106°	-6°	
Salt Lake City.....	102°	-20°	
St. Paul.....	104°	-41°	

In finding the range of temperature and the difference in altitude you have really had to subtract a negative number from a positive number. Instead of saying "Subtract -32 from 110 ," we may say "What is the difference between 110 and -32 ?" or "What number added to -32 makes 110 ?" Whichever way the question is asked the answer is most easily understood if one turns to the number scale and asks himself what is the difference or distance between 110 and -32 . The answer is 142 . If a lower number on the scale is subtracted from a higher number, the answer is positive; if a higher number is subtracted from a lower, the answer is negative.



EXERCISE 38

1. The minimum temperature in Seattle is how much above the minimum in Omaha? (To subtract -32 from $+110$, how far is it on the number scale from -32 to $+110$?)

2. The minimum in St. Paul is how much lower than the minimum in Block Island? State a sentence comparing the maximum in these two places.

3. Subtract -41 from -6 . What number added to -10 makes -3 ?

4. How many degrees north must a vessel sail from Rio de Janeiro, 23° south latitude, to New York, 41° north latitude?

5. What number added to -23 makes 41 ? Subtract -23 from 0 .

6. A vessel sailing 12° south from Rio de Janeiro reaches the latitude of Buenos Aires. What is the latitude of the latter city?

7. Subtract 12 from -23 . From 12 subtract -23 .

8. How many degrees of latitude separate New York from Buenos Aires? Allowing 69 miles to a degree, how many miles to the southward is Buenos Aires?

9. If east longitude is considered positive, how can you write the longitude of Cape Horn, 67° west longitude?

10. How many degrees of longitude separate Cape Horn from the Cape of Good Hope, 18° east longitude? Subtract -67 from 18 .

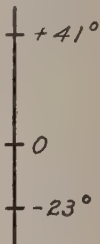
11. The senior class of the Dover High School started on a trip to Washington with $\$128$ and came home $\$13$ in debt. How much was spent by the class?

12. The Woolworth Building is $792'$ high, above the sidewalk, and the foundation reaches $115'$ below the sidewalk. Find the distance from the bottom of the foundation to the top of the building.

13. A coal dealer has on hand 70 tons of coal; he has unfilled orders amounting to 135 tons. How many tons is he short? Subtract 135 from 70 .

14. A rower makes 4 miles an hour in still water. How fast can he row downstream if the current flows at the rate of 3 miles an hour? How fast can he row upstream?

NORTH

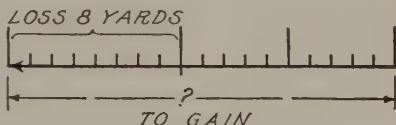


15. Henry Sullivan, who swam across the English Channel, could swim 2 miles an hour. How fast could he approach the French shore when the tide was going out $1\frac{1}{2}$ miles an hour? When the tide was going out $2\frac{1}{2}$ miles an hour? (Note: Sullivan kept on swimming until the tide turned in.)

16. What would be the meaning of a negative answer to each of these questions:

- (a) How fast is the rowboat going upstream?
- (b) What is Mr. Smith's bank balance?
- (c) How many degrees north latitude is the place?
- (d) How many pounds have you gained in weight this year?

17. The Yale football team in three plays lost a total of 8 yards. What distance was there to gain on the fourth down?



(In football 10 yards must be gained in four plays or "downs.")

18. The Princeton team gained 9 yards in two plays but on the third down were penalized 15 yards. What distance was there to gain on the fourth down?

19. Add $+9$ and -15 . Subtract -6 from $+10$.

20. The Michigan team in four plays gained 6 yards, lost 8 yards, lost 3 yards, gained 14 yards. Did they make the required distance of 10 yards? (If not, the opposing side receives the ball, and tries four plays.)

21. Dartmouth lost 5 yards on the first down, gained 11 yards on the second down, lost 2 yards on the third down, and gained 7 yards on the fourth down. Did Dartmouth keep the ball?

22. On the first down the Iowa quarter back fumbled the ball and 10 yards were lost, but on the second down a forward pass gained 25 yards. As a combined result of the two plays, what progress was made?

23. Subtract -3 from $+10$. Add 25 to -10 .

24. Felton of Harvard, standing 8 yards behind his goal line, punted the ball 50 yards ahead of him from where he stood. How far did the ball go in front of the goal line?

25. Add 50 to -8 .

26. Catching the ball 4 yards behind his goal line the West Point full back ran out with the ball to a line 25 yards in front of it. How many yards did he run?

27. Mr. White owes $\$9000$ but has assets of $\$12,000$. What are his net resources?

28. Mr. Black owes $\$15,000$, but has assets of $\$11,000$. What are his net liabilities? Is there any way to express his net resources?

29. If an agent loses $\$650$ on one sale and makes $\$720$ on another, what is the net result?

30. Add -650 to $+720$.

31. Add 32 to -40 . Subtract 17 from -10 .

In many formulas and equations we have negative values for some of the quantities, and our results themselves are often negative numbers.

Example: Find the value of $M + 5$ if $M = -8$.

Solution:

The expression $M + 5$ means that M and 5 are to be added. Add -8 and 5 by starting at -8 and counting up 5 , getting -3 .

Briefly, if $M = -8$, $M + 5 = -8 + 5 = -3$ Ans.

EXERCISE 39

1. What is the value of $a + b$ if $a = -5$, $b = +10$? if $a = -6$, $b = +6$?

2. What is the value of $a - b$ if $a = 7$, $b = 10$? if $a = -5$ and $b = 3$?

3. Add 5 and 7.

4. Add -5 and -7 .

5. Add 7 to -5 .

6. What is the sum of $+10$ and -10 ?

7. To -10 add $+13$.

8. To -11 add $+8$.

9. Add $6M$ and $+3M$.

10. Find the sum of $-9x$ and $+3x$.

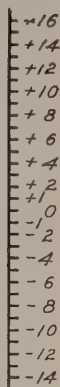
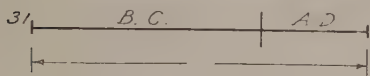
11. Add $+4p$ to $-4p$.

12. Augustus Caesar became the first emperor of

Rome in 31 B.C.

and ruled 45 years.

When did he die?



13. What is $-31 + 45$? $-21 + 15$?

14. Alexander the Great lived only 13 years after coming to the throne in 336 B.C. What was the date of his death?

15. Mr. Grant owed me \$15. Today he paid me \$10. What is his debt to me now?

16. Add 13 to -336 . Add 15 and -10 .

17. What is the average of 15, 25, and 8?

18. What is the average of -3 , $+7$, and -1 ?

19. Find the average temperature for the January day on page 76.

20. The Romans dated from the founding of Rome in 753 B.C. If the Roman dates were still in use what would be the number of the present year? What was the Roman date for the murder of Julius Caesar, 44 B.C.?

21. Subtract -753 from $+1000$. Subtract -753 from -44 .

22. Find the value of $p - c$, if $p = \$65$ and $c = \$50$; if $p = 40$ cents and $c = 48$ cents.

23. What number added to 8 would make 15? What number added to 15 would make 8?

24. Supply the number missing:

(a) $6 + ? = 10$

(b) $? + 15 = 9$

(c) $-10 + ? = 0$

(d) $? + 4 = -3.$

25. What value of x would make each equation true?

(a) $8 + x = 12$

(b) $x + 12 = 7$

(c) $-6 + x = 6.$

26. 8 and how many are 12?

27. What number added to 8 would make -12 ?

28. What number added to -9 would make $+14$?

29. What number added to -6 would give -6 ?

30. Subtract 0 from -5 .

31. From $4N$ take $7N$.

32. What is the difference between $+10$ and -13 ?

33. Subtract 5 from 0.

34. $9 - 14 = ?$

35. How would you interpret the meaning of a negative answer if you were finding:

(a) How many years older is John than Alec?

(b) How many dollars a clothing store gained on an overcoat?

(c) The number of degrees east longitude of a ship's location at sea?

(d) The increase in the number attending your school?

CHAPTER V

ADDITION AND SUBTRACTION

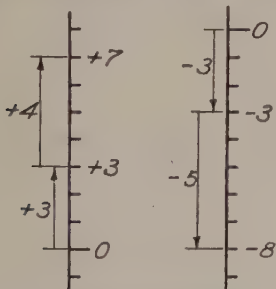
ADDITION

The addition of the two positive numbers $+3$ and $+4$ is illustrated by the diagram of the number scale below; the sum is $+7$.

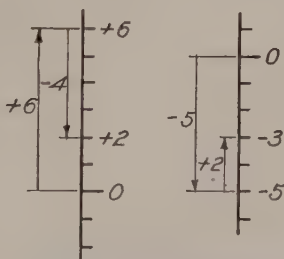
To add two negative numbers such as -3 and -5 with the help of the number scale we count down 3, and starting from -3 we count down 5; the sum is -8 .

The absolute value of any number is obtained by disregarding the sign of the number. Thus $+5$ and -5 have the same absolute value, 5. What is the absolute value of -3 ? $+4$? -7 ? $+3$?

A rule for finding the sum of any two numbers having like signs may now be stated: *To add any two numbers having like signs find the sum of the absolute values and prefix the common sign.*



ADDING TWO NUMBERS
HAVING LIKE SIGNS.



ADDING TWO NUMBERS
HAVING UNLIKE SIGNS.

The addition on the number scale of two numbers having unlike signs is illustrated above. The sum of $+6$ and -4

is shown to be $+2$, and that of -5 and $+2$ is shown to be -3 . This suggests the following rule for finding the sum of two numbers having unlike signs: *To add any two numbers having unlike signs find the difference of the absolute values and prefix the sign of the greater.*

• EXERCISE 40

Add

$$\begin{array}{r} 1. \quad +5 \\ \quad +7 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad +17 \\ \quad -6 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad -4m \\ \quad -3m \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad -5 \\ \quad -7 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad -9 \\ \quad +9 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad +7D \\ \quad -2D \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad -8 \\ \quad -1 \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad -1.5 \\ \quad +2.5 \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad +9bc \\ \quad -13bc \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad -10 \\ \quad +3 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad +2a \\ \quad +3a \\ \hline \end{array}$$

$$\begin{array}{r} 12. \quad +x \\ \quad -x \\ \hline \end{array}$$

Frequently the numbers to be added are found horizontally arranged. They are to be added mentally without copying them in columns.

$$13. \quad +7, -2$$

$$17. \quad -2w - 4w$$

$$14. \quad -10, +18$$

$$18. \quad +3z - 10z$$

$$15. \quad -11x^2, -5x^2$$

$$19. \quad 13xy - 20xy$$

$$16. \quad -3 + 8$$

$$20. \quad R - 4R$$

Observe that sometimes the plus sign is not written when the first number given is positive.

When there are several numbers to be added, some of them positive, some negative, it is a principle of the order of operations that the numbers may be added in any order. It is usually most convenient first to add all the positive numbers,

then to add all the negative numbers, and finally to combine the two results.

Example 1:

Add $+7, -3, +5, -8$

$$\begin{aligned}\text{Solution: } 7 - 3 + 5 - 8 &= 7 + 5 - 3 - 8 \\ &= 12 - 11 \\ &= 1 \quad \text{Ans.}\end{aligned}$$

Example 2:

Add $x + 7x - 6x - 10x + 3x$

Solution:

$$\begin{aligned}x + 7x - 6x - 10x + 3x &= x + 7x + 3x - 6x - 10x \\ &= 11x - 16x \\ &= -5x \quad \text{Ans.}\end{aligned}$$

In practice we do not actually rewrite the numbers in a changed order, but merely add them in that order mentally.

Example 3:

Add $10an$

$-5an$

$+an$

$-3an$

$+3an$

We think mentally $11an$ and $-8an$ and add the coefficients 11 and -8 .

EXERCISE 41

Add:

$$\begin{array}{r} 1. \quad +10 \\ \quad -3 \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad +5 \\ \quad +1 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad -1 \\ \quad -1 \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad +7m \\ \quad -7m \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 4xy \\ \quad 5xy \\ \quad 6xy \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad -3m \\ \quad +8m \\ \quad -10m \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad +5\pi \\ \quad -2\pi \\ \quad +5\pi \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 3 \times 37 \\ \quad 2 \times 37 \\ \quad 5 \times 37 \\ \hline \end{array}$$

$$9. \quad 5 + 9 - 4 - 3$$

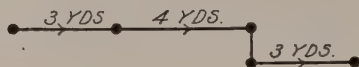
$$12. \quad 5a - a + 12a$$

$$10. \quad -6 + 10 - 8 + 2$$

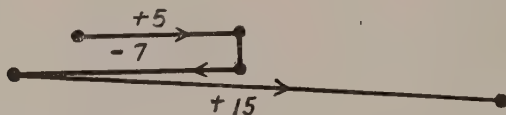
$$13. \quad 4x^2 + 3x^2 - 5x^2$$

$$11. \quad R + 2R + 3R$$

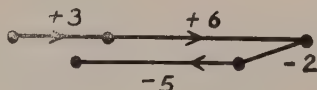
14. The changes in price of the shares of six railroads in the stock market on a certain day was $+4\frac{1}{2}$, -2 , $+\frac{3}{4}$, $-\frac{1}{8}$, $+1$, $-\frac{1}{2}$. What was the average change in prices?



15. The Georgia team, in four plays, advanced the ball 3 yards, 4 yards, 0 yards, 3 yards. What was the total advance?



16. The Oregon team, in successive plays, gained 5 yards, gained 0 yards, lost 7 yards, gained 15 yards. What was the total gain?



17. In four successive plays Cornell advanced the ball in yards $+3$, $+6$, -2 , -5 . The total gain was how many yards?

18. The profit from each day's business of the boat *Plymouth* was \$75, \$50, $-\$20$, $-\$40$, $+\$10$, $+\$100$, $+\$150$. What was the profit for the week? What was the average profit per day?

When the numbers to be added are not all of the same kind we usually arrange in columns, as in arithmetic. We must make sure that all the terms in each column are like terms, differing only in their numerical coefficients.

Example 1:

Add 2 ft., 4 in., 3 ft., 2 in.
and 4 ft. 3 in.

$$\begin{array}{r} 2 \text{ ft. } 4 \text{ in.} \\ 3 \text{ ft. } 2 \text{ in.} \\ 4 \text{ ft. } 3 \text{ in.} \\ \hline 9 \text{ ft. } 9 \text{ in.} \end{array}$$

Example 2:

Add $2f + 4i$, $3f + 2i$,
and $4f + 3i$.

$$\begin{array}{r} 2f + 4i \\ 3f + 2i \\ 4f + 3i \\ \hline 9f + 9i \end{array}$$

Example 3:

Add $6b + g$, $5b - 3g$, and $-4b + 8g$. Check when $b = 1$, $g = 1$.

Solution:

$$\begin{array}{r} 6b + g \\ 5b - 3g \\ -4b + 8g \\ \hline 7b + 6g \end{array}$$

Example:

$$\begin{array}{r} 6 + 1 = 7 \\ 5 - 3 = 2 \\ -4 + 8 = 4 \\ \hline 13 \end{array}$$

Check:

Answer:

$$\begin{array}{r} 7b + 6g \\ 7 + 6 = 13 \end{array}$$

$$\overline{13} \longleftrightarrow \overline{13}$$

EXERCISE 42

Copy and add:

1. $\begin{array}{r} 341 \\ 527 \\ \hline \end{array}$

2. $\begin{array}{r} 3h + 4t + u \\ 5h + 2t + 7u \\ \hline \end{array}$

3. $\begin{array}{r} 4 \text{ lb. } 5 \text{ oz.} \\ 3 \text{ lb. } 2 \text{ oz.} \\ 1 \text{ lb. } 6 \text{ oz.} \\ \hline \end{array}$

4. $\begin{array}{r} 4l + 5h \\ 3l + 2h \\ l + 6h \\ \hline \end{array}$

5. $\begin{array}{r} 15 \text{ ft. } 7 \text{ in.} \\ 6 \text{ ft. } 1 \text{ in.} \\ 13 \text{ ft. } 2 \text{ in.} \\ \hline \end{array}$

6. $\begin{array}{r} 15f + 7i \\ 6f + 5i \\ 13f + 4i \\ \hline \end{array}$

7. $\begin{array}{r} 2 \text{ bu. } 1 \text{ pk.} \\ 4 \text{ bu. } 3 \text{ pk.} \\ 10 \text{ bu. } 2 \text{ pk.} \\ \hline \end{array}$

8. $\begin{array}{r} 4a^2 + a \\ 10a^2 + 10a \\ a^2 + 2a \\ \hline \end{array}$

9. 4.6

1.732

3.142

2.05

10. $5x + 6y$

$3x + 7y + 2$

$4x + y$

$10x \quad + 6$

11. Add \$5.23, \$7.08, \$15.90, \$8, \$23, and \$10.05.

12. Add 6.75, .1429, 13.5, 40.083, and .0625.

In the remaining examples check when each letter equals 1:

13. $6m - 4$

$7m - 2$

17. $2a - 3b + 4$

$a \quad + 1$

$4a + 2b - 3$

14. $3x + 9y - 7$

$4x + 6y - 5$

18. $5r - s - 1$

$- 7r + s + 10$

$- 2r + 4s$

15. $5l + 4w$

$2l + 2w$

$3l - 3w$

19. $-3x + 5y + z$

$x - 2y + 4z$

$4x - 3y - z$

16. $4x^2 + 5x - 3$

$5x^2 + 2x - 10$

$12x^2 - 9x + 2$

20. $x^2 - xy + y^2 - 1$

$x^2 + 2xy + y^2 - 9$

$x^2 \quad - y^2$

Arrange like terms in columns, add, and check:

21. $2y + 3, 4y - 2, y + 10$

22. $m^2 + 5n^2, +4m^2 - 2n^2, -3m^2 + 10n^2$

23. $3a - 2b, +7b + 9a, -3b$

24. $x + 9y - 5 + 3x - 4y + 7 - 6y + x - 2$

When there are different powers of the same letter it is convenient to arrange the terms in order of the descending powers of that letter, such as x^3, x^2, x .

$$25. \quad 2r + 5r^2 - 3 + r^2 + 2r + 1 - 12 - 3r^2 + 12r$$

Solution:

Arranging in descending powers, r^2 being the highest power, we get

$$\begin{array}{r} + 5r^2 + 2r - 3 \\ + r^2 + 2r + 1 \\ - 3r^2 + 12r - 12 \\ \hline + 3r^2 + 16r - 14 \end{array} \quad \text{Ans.}$$

Check, when $r = 1$

Example:

Answer:

$$\begin{array}{r|l} 5 + 2 - 3 = 4 & + 3r^2 + 16r - 14 \\ 1 + 2 + 1 = 4 & 3 + 16 - 14 \\ - 3 + 12 - 12 = -3 & \\ \hline & +5 \leftarrow \longrightarrow +5 \end{array}$$

$$26. \quad h^3 - 8 + h^2 - 4h + 4 - 3h^3 + 6h$$

$$27. \quad 3ab + a^2 - b^2 - a^2 + 2ab - b^2 + 4b + 4a^2$$

$$28. \quad 6yz^2 - 4y^2z + z^3 - 3y^2z + 3yz^2 - z^3 + y^3$$

Add the following without arranging in columns, and check:

$$29. \quad 2l + 3s + 5s - 6l$$

$$30. \quad p + q - p - q + p$$

$$31. \quad 3x - 3 - 2 + x^2 - 2x + 1$$

$$32. \quad -x + 3y + 8 + 5y - 2x + y - 20$$

$$33. \quad n^2 - 9 - 2n + 10 - 3n^2 + 4n - 1$$

$$34. \quad x - 2d, x - d, x, x + d, x + 2d$$

Express in the briefest way, with the fewest possible terms, and check:

$$35. \quad l + l - 15 + l + l - 15$$

$$36. \quad 10\pi - 3\pi + 2h - 7\pi + 4h$$

$$37. \quad 5m - 2 - 10x + 7 - m + x - 5$$

$$38. \quad a^2 + 2ab + b^2 - a^2 + 2ab - b^2 - a^2 + b^2$$

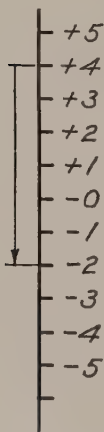
39. If $P = 2L + 2w$ and $P' = 6L + 4w$ find the value of $P + P'$.

40. If $y = 7x - 17$ and $z = 8x + 11$ find the value of $y + z$.

41. If $a = n^2 - n - 2$, $b = n^2 - 2n + 1$, and $c = 4 - n^2$ find the value of $a + b + c$.

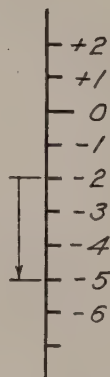
42. Can the sum of two algebraic numbers be lower on the number scale than one of the numbers? than either of them? Give examples.

SUBTRACTION



From 4 subtract 6.

Ans. -2



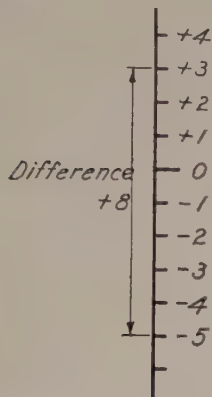
From -2 subtract 3.

Ans. -5

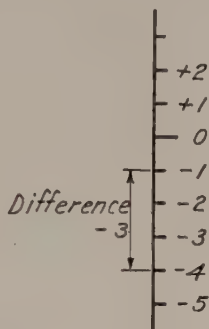
To subtract a positive number, count down on the number scale as in the two examples above.

To subtract a negative number ask what must be added to the subtrahend to make the minuend, or how many spaces on the number scale separate the subtrahend from the minuend. Whenever a number is subtracted from one which is

lower than itself on the number scale, the difference is negative



Subtract -5 from $+3$



Subtract -1 from -4

Read each of the following subtraction examples, using the words "What number added to — gives —?" After answering the question, check by considering the number scale.

EXERCISE 43

Solve the first example by asking the question, "What is the number which added to $+7$ gives $+10$?"

1. $+10$	2. $+5$	3. $+4$	4. -2	5. -3
$+7$	$+7$	-3	-2	$+3$

6. $5a$	7. $2m$	8. $-4x^2$	9. $-5yz$	10. $-6R^2$
$2a$	$5m$	$+2x^2$	$-8yz$	$-2R^2$

11. What number added to -4 gives 0? Subtract -4 from $+4$.

12. If you owe a debt of \$5 and someone pays it for you, are you richer or poorer?

13. By diminishing one's debts are one's resources increased or decreased?

An example such as $7 - 5$ may be considered as the subtraction of 5 from 7 or as the addition of -5 to 7. The subtraction of any number is equivalent to the addition of a number having the same absolute value but the opposite sign. The subtraction of -4 from $+6$ gives the same result as the addition of $+4$ to $+6$. Taking away a debt of \$5 really adds \$5 to one's resources. A rule for subtraction is therefore: *Change the sign of each term of the subtrahend and add.* Do not change the sign in the work on your paper, but change it mentally.

When the minuend or subtrahend contains more than one term the work is arranged in columns of like terms, as it was for addition.

Example 1: Subtract $4b - 2$ from $5b + 3$.

Solution:

$$\begin{array}{r} \text{Minuend} \quad 5b + 3 \\ \text{Subtrahend} \quad 4b - 2 \\ \hline \text{Difference} \quad b + 5 \end{array}$$

Check, when $b = 1$:

Example:	Answer:
$5 + 3 = 8$	$b + 5$
$4 - 2 = 2$	$1 + 5 = 6$
6	$6 \longleftrightarrow 6$

Example 2: From $3x^2 - x - 5$ subtract $x^2 - 4x - 2$

Check, when $x = 1$:

Solution:

$$\begin{array}{r} 3x^2 - x - 5 \\ x^2 - 4x - 2 \\ \hline 2x^2 + 3x - 3 \end{array}$$

Example:

$$\begin{array}{r} 3 - 1 - 5 = -3 \\ 1 - 4 - 2 = -5 \end{array}$$

Answer:

$2x^2 + 3x - 3$	$2x^2 + 3x - 3$
$2 \quad + 3 \quad - 3 =$	$2 \quad + 3 \quad - 3 =$
$+2$	$+2 \longleftrightarrow +2$

EXERCISE 44

1. Subtract 4 from 1. Add -4 and $+1$.
2. Work these examples after rewording each as a question beginning "What number added to ——?" Work them

again by changing the sign of the subtrahend, and compare answers.

- (a) Subtract -5 from -2 .
- (b) From 3 subtract -4 .
- (c) Subtract $-10x$ from $-3x$.
- (d) Find the difference between $-2ab$ and $+2ab$.

Subtract:

$$\begin{array}{r} 3. \ 347 \\ \underline{135} \end{array}$$

$$\begin{array}{r} 4. \ 3t^2 + 4t + 7 \\ \underline{t^2 + 3t + 5} \end{array}$$

$$\begin{array}{r} 5. \ 5 \text{ ft. } 7 \text{ in.} \\ \underline{3 \text{ ft. } 4 \text{ in.}} \end{array}$$

$$\begin{array}{r} 6. \ 5f + 7i \\ \underline{3f + 4i} \end{array}$$

$$\begin{array}{r} 7. \ 14 \text{ bu. } 3 \text{ pk.} \\ \underline{2 \text{ bu. } 1 \text{ pk.}} \end{array}$$

$$\begin{array}{r} 8. \ 14b + 3p \\ \underline{2b + p} \end{array}$$

$$\begin{array}{r} 9. \ 4.07 \\ \underline{3.585} \end{array}$$

$$\begin{array}{r} 10. \ 13.6 \\ \underline{7.854} \end{array}$$

$$\begin{array}{r} 11. \ \text{Yr.} \quad \text{Mo.} \quad \text{Day} \\ 1924 \quad 12 \quad 21 \\ \underline{1921 \quad 2 \quad 13} \end{array}$$

$$\begin{array}{r} 12. \ \text{Yr.} \quad \text{Mo.} \quad \text{Day} \\ \quad \quad \quad (\text{Fill in today's date}) \\ \underline{1776 \quad 7 \quad 4} \end{array}$$

13. Subtract 3.14159 from 10.

14. What is the difference between 12.5664 and 8.7?

Subtract, and check when each letter equals 1:

$$\begin{array}{r} 15. \ 10a + 2 \\ \underline{6a - 1} \end{array}$$

$$\begin{array}{r} 18. \ 3L - w \\ \underline{2L + 2w} \end{array}$$

$$\begin{array}{r} 16. \ 5x - 3y \\ \underline{2x - y} \end{array}$$

$$\begin{array}{r} 19. \ 4\pi R + \pi R^2 \\ \underline{2\pi R + \pi R^2} \end{array}$$

$$\begin{array}{r} 17. \ 8b + 5c \\ \underline{2b + 7c} \end{array}$$

$$\begin{array}{r} 20. \ 3x + 5y - 6 \\ \underline{2x - 3y + 2} \end{array}$$

$$21. \frac{x^2 - xy + y^2}{x^2 - y^2}$$

$$23. \frac{-m^2 + mn + 12n^2}{-3m^2 + 2mn + 5n^2}$$

$$22. \frac{x^2 - 4a^2}{x^2 + 2ax - a^2}$$

$$24. \frac{a^3 - 6a^2b + 12ab^2}{a^3 - 3a^2b + 3ab^2 - b^3}$$

25. Subtract $x + 3y - 2$ from $2x - y + 5$.

26. From $3x^2 + 2x - 7$ subtract $3x - 7 - x^2$.

27. Take $a + b - c$ from $a + c - b$.

28. From $x - y$ take $x + y$.

29. What is the difference between $a^2 + 2ab + b^2$ and $a^2 + b^2$?

30. Find the difference between $r^2 - 2rh + h^2$ and $r^2 - rh - 2h^2$.

31. What number added to $nd - d + 2f$ would give $d + 2f$?

32. Subtract $a + 2b - 3c$ from 0.

33. If $P_1 = 2L + 2w$ and $P_2 = 2L - 4w$ find the value of $P_1 - P_2$.

34. If $x = 3a - 2b$ and $y = 3b - 2a$ find the value of $x - y$.

35. By how much does seventeen and six hundredths exceed six and seven tenths?

36. From the sum of $Ax + By + C$ and $Ax - By + C$ subtract $Ax - C$.

37. From the sum of $a^2 - 4ab + 4b^2$ and $4a^2 - b^2$ take the sum of $a^2 - 9b^2$ and $12b^2 - a^2$.

38. Subtract the sum of $x - y$ and $y - z$ from $3x - y + z$.

Parentheses. Signs of grouping are the parenthesis (), the bracket [], and the brace { }. These signs are all spoken of as parentheses. They may be used interchangeably.

Since an expression within a parenthesis is to be considered as a unit, it is a rule of the order of operations that all opera-

tions within a parenthesis are to be performed first, before considering what is outside the parenthesis.

Example 1: Find the value of $(10 - 3) - (4 + 1)$.

$$\begin{aligned}(10 - 3) - (4 + 1) &= \\ 7 - 5 &= 2 \quad \text{Ans.}\end{aligned}$$

Parentheses are frequently used to denote additions and subtractions, doing away with many words by the use of algebraic symbols.

Example 2: Express without words this example: "Subtract the difference between a and b from c ."

$$\text{Ans.: } c - (a - b)$$

When a parenthesis is preceded by a minus sign, a subtraction is to be performed, and one may perform the subtraction by changing each sign of the subtrahend, which is the quantity inside the parenthesis, and afterwards adding.

Example 3: Simplify $7 - (5 - 3)$.

Solution: $7 - (5 - 3)$.

Changing the signs of the subtrahend, $7 - 5 + 3$

Adding, 5 Ans.

Check:

Example:	Answer:
$7 - (5 - 3) =$	
$7 - 2 =$	
5	5
	←————→

Example 4: $(a + b) - [(a - b) - (c + d)]$.

Solution: First work with the inner parentheses, changing signs of the subtrahend $(c + d)$,

$$(a + b) - [a - b - c - d]$$

Next change signs of the subtrahend in the bracket,

$$a + b - a + b + c + d$$

Adding $2b + c + d$ Ans.

To simplify expressions containing parentheses:

1. First consider the inner parentheses,—*i.e.* those which do not themselves include any other parenthesis.
2. A parenthesis preceded by a plus sign may be removed without changing the terms within.
3. A parenthesis preceded by a minus sign is a subtrahend, and the parenthesis may be removed if the sign of each term within is changed.
4. Remove any parentheses that remain.
5. When all parentheses have been removed, collect like terms (*i.e.* add).

EXERCISE 45

1. Simplify $(7 + 2x) - (5 - 3x)$.
2. Write Example 1 in column form as a subtraction, and subtract.
3. (a) Subtract $5a - 3$ from $9a - 1$.
(b) Write this example with parentheses, and simplify.
4. Simplify $(x - 2y + 3) - (2x - y - 2)$ and check by doing the example as a subtraction.
5. Simplify $25 - (6 + 10)$ and check as in model Example 3.
6. Find the value of $(35 - 2) - (7 - 3)$ and check.
7. Find the value of $(10 - 1) + (1 - 10) - (10 + 1)$.
8. Indicate by symbols only "Add $3x + 5$ and $6 - 2x$."
Answer to the addition = ?
9. Express without words "From $3x - y + 1$ subtract $x + 3y - 2$." Simplify.

Do each of the following examples twice; first, by removing the parentheses and simplifying; second, by a subtraction in column form:

10. $(5m + 1) - (2m + 1)$
11. $10x - (3x - 1)$

12. $6y - (5x + y)$
13. $(14b - 3c) - (b - c)$
14. $5 - (2x + 1)$
15. $90 - (90 - n)$
16. $17 - a - (a - 17)$
17. $(a + b) - (a - b)$
18. $x + 10y - (7x - 2y - 1)$
19. $(x^2 + 2x - 15) - (9 - 6x + x^2)$

Simplify and check by substituting 1 for each letter:

20. $(10k + 1) + (k + 2) - (k + 3)$
21. $a - (a - x) + (a - 2x)$
22. $p - (q - r) - (r + q)$
23. $n - (10 - l - 10)$
24. $5 - [6 - (7 - m) + 4]$ Ans. $2 - m$
25. $[x + (a - b)] - [x - (a + b)]$
26. $a - [b - (c - d)] + d - [a - (b + c)]$

Occasionally it is necessary to insert parentheses, and this may be easily done if we understand how to remove them. Terms may be included in a parenthesis preceded by a plus sign without any change; terms may be enclosed in a parenthesis preceded by a minus sign if we change the sign of each term so enclosed.

Example: Enclose the last three terms of $x^3 - 3x^2 + 3x - 1$ in a parenthesis preceded (a) by a plus sign; (b) by a minus sign.

Solution:

$$(a) \quad x^3 - 3x^2 + 3x - 1 = \\ x^3 + (-3x^2 + 3x - 1)$$

$$(b) \quad x^3 - 3x^2 + 3x - 1 = \\ x^3 - (3x^2 - 3x + 1)$$

EXERCISE 46

Enclose the last two terms of each expression (a) in a parenthesis preceded by a plus sign, (b) in one preceded by a minus sign:

1. $3 + 2 - 1$

4. $2f + nd - d$

2. $a - b + c$

5. $p - q + r - l$

3. $5x - 3x^2 - 2x^3$

6. $x + y + z$

Enclose the last three terms of each expression in a parenthesis preceded by a minus sign:

7. $4 - 5 + 6 - 7$

10. $7 - 6m + 5n - 4 + p$

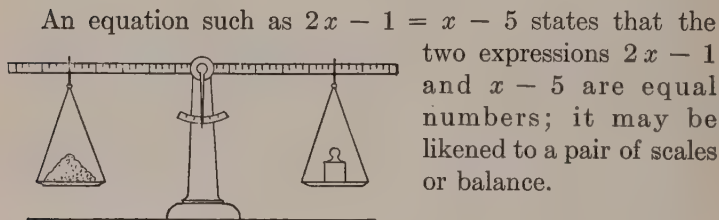
8. $a + b - c - d$

11. $x^2 - a^2 + 2ab - b^2$

9. $-h - k - l - m$

12. $1 + r + r^2 + r^3$

EQUATIONS



EXERCISE 47

1. An unknown weight of w ounces is placed in one pan of the scales. If it balances a 4-ounce weight in the other pan, what is w ?

2. A dealer has lost his 8-ounce weight but has still the 16, 4, 2, and two 1-ounce weights. How much is w if together with the 4-oz. weight it balances the 16-oz. weight?

3. Solve mentally the equation $w + 4 = 16$.

4. What is w if together with the 1-oz. weight it balances the 16-oz. weight?

5. Solve mentally $w + 1 = 16$

6. Find w for each balance:

In one pan,—

(a) $w + 1$

(b) $w + 4 + 1$

(c) $w + 2 + 4$

In the other pan,—

4

16

16

7. How could the dealer arrange the weights (see Example 2) to find w if w were 13 ounces? 9 ounces?

8. Could you arrange the weights so as to weigh any amount from 1 ounce to 24 ounces?

If one adds the same number of ounces to both pans of the scale, or if one takes away the same amount, the balance will not be disturbed.

Likewise, *the same number may be added to each side of an equation, or the same number may be subtracted from each side, without affecting the truth of the equation.* To keep the balance which exists between the two sides of an equation no change of any kind must be allowed to take place in the value of one side without a like change on the other side.

These facts suggest three fundamental principles, known as axioms:

Axiom I. If equal numbers are added to equal numbers, the sums are equal.

Axiom II. If equal numbers are subtracted from equal numbers, the differences are equal.

Axiom III. If equal numbers are divided by equal numbers (excepting zero), the quotients are equal.

The third axiom has been used in a previous chapter when we solved equations like $7x = 42$ by dividing each side by the coefficient 7.

THE USE OF THE SUBTRACTION AXIOM

Many equations can be solved with the help of the second axiom, by subtracting the same number from each side of the equation.

Example: Solve $2w + 3 = 7$

Solution:

$$2w + 3 = 7$$

Subtracting 3 from each side, $2w = 4$

Dividing each side by 2, $w = 2$. Ans.

Check:

The left side

The right side

$$2w + 3 =$$

$$2(2) + 3 =$$

$$4 + 3 =$$

$$7 \longleftarrow \longrightarrow 7$$

EXERCISE 48

Write out all the steps in the solution of each example, arranging your work as in the example above:

1. A boy with 8 pounds of clothes on him weighs 105 pounds. What does the boy himself weigh? (Solve $w + 8 = 105$.)

2. A baby is placed in a basket and the weight is found to be 20 pounds. The basket weighs 3 pounds when empty. What does the baby weigh? (Write an equation.)

3. An empty box weighs 2 ounces. When filled with chocolates it weighs 1 pound. What is the weight of the chocolates? (Equation?)

Solve by subtracting a number from each side:

4. $x + 5 = 12$

7. $7n + 4 = 25$

5. $n + 10 = 95$

8. $11y + 1 = 100$

6. $3x + 8 = 32$

9. $3x + \frac{1}{2} = 8$

10. $\frac{1}{2}x + 9 = 13$

11. $2.5x + .75 = 7$

12. $3x = x + 8$ Hint: Subtract x from each side

13. $4n = 2n + 10$

14. $15x = 25 + 10x$

15. $n + n + 1 = 29$

16. $3x - x = x + 4$

17. $5r = 2r + 16 - r$

Express as equations, and solve:

18. What number (n) added to 7 will give 16?

19. Twice a certain number is equal to 12 added to the number. What is the number?

20. Three times my age is 28 years more than my age. How old am I?

21. I am thinking of a number which if doubled and then increased by 4 would make 50. What is the number?

THE USE OF THE ADDITION AXIOM

Some equations may be solved with the help of the first axiom by adding the same number to each side of the given equation.

Example 1: Solve $5n = 14 - 2n$.

Solution:

$$5n = 14 - 2n$$

Adding $2n$ to each side, $7n = 14$.

Dividing each side by 7, $n = 2$. Ans.

Check:

The left side

The right side

$$5n =$$

$$5(2) =$$

$$10$$

$$14 - 2n =$$

$$14 - 2(2) =$$

$$14 - 4 =$$

$$10$$

When there are several terms in an equation, successive additions or subtractions may be needed.

Example 2: Solve $21 - 3x = 31 - 5x$.

Solution: $21 - 3x = 31 - 5x$

Subtracting 21 from each side, $-3x = 10 - 5x$

Adding $5x$ to each side, $2x = 10$

Dividing each side by 2, $x = 5$ Ans.

Check:

The left side

$$21 - 3x =$$

$$21 - 15 =$$

The right side

$$31 - 5x =$$

$$31 - 25 =$$

6 $\longleftrightarrow$ 6

EXERCISE 49

Write out all the steps of each solution, as in Examples 1 and 2 above:

1. $x - 6 = 7$

2. $n - 10 = 10$

3. $3y - 1 = 5$

4. $7n - 3 = 11$

5. $9x = 20 - x$

6. $2n - 10 = -4$

7. $5x + 6 = x + 2$

8. $8r + 3 = 13 - 2r$

9. $4y - 3 = y - 9$

10. $5w + 2 = 30 - 2w$

11. $10 - (2 - x) = 12$

12. $3x = 17 - (x + 1)$

13. $2n - (n - 1) = 8$

Express each problem in equation form, and solve:

14. If 12 is subtracted from a certain number the difference is 10. Find the number.

15. Five times a certain number exceeds 20 by three times the number. What is the number?

16. If 6 is taken from three times a certain number the difference is 6. Find the number.

17. Florence said that if she doubled her age and added 19 the sum would be 47. How old is Florence?

18. Four Generalissimo cigars cost 24 cents more than one cigar. What does one cigar cost?

19. To what number must 16 be added if the result is to be 10 less than twice the number?

20. What number is as much more than 10 as twice the number is less than 27?

21. Carl thought of a number, doubled it, added 17, and said he obtained the same sum as if he had added 28 to the number. What was the number?

22. How can a stick 36" long be cut in two so that one part will be seven times as long as the other?

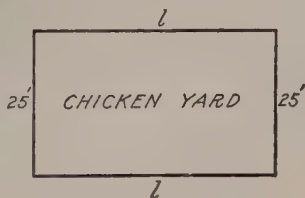
23. Each arm of an isosceles triangle is double the base, and the perimeter is 70 inches. What is the base?

24. The base of an isosceles triangle is 14" and the perimeter is 40". How long is an arm?

25. If the perimeter of an isosceles triangle is 50' and an arm is 16', find the base.

26. The perimeter of a rectangle is 42 inches and the length is 12 inches. What is the width?

27. A farmer wishes to enclose a yard for his chickens, the yard to be 25 feet wide. He has 130 feet of wire fence and wishes to use it all, making the yard as long as possible. How long can it be?



CHAPTER VI

MULTIPLICATION AND DIVISION

THE PRODUCT OF A POSITIVE AND A NEGATIVE NUMBER

EXERCISE 50

1. How could a person who knows how to add, but knows nothing of multiplication, find the cost of 3 articles at 17 cents apiece?

2. By adding, find the product of 3 times 23; of 3 times $2a + 3x$.

3. Add to get the perimeter of a square, side 3 ft. 2 in.

4. Find the product of 2 times -13 by adding.

5. To find the product $5(2a - 3b)$ we may set down $2a - 3b$ five times, and add:

$$\begin{array}{r} 2a - 3b \\ 2a - 3b \\ 2a - 3b \\ 2a - 3b \\ 2a - 3b \\ \hline 10a - 15b \end{array}$$

$$\therefore 5(2a - 3b) = 10a - 15b \quad \text{Ans.}$$

6. By adding, multiply $6x^2 + 2x - 5$ by 4.

7. In the above examples what is the product of 2 and -13 ? of 5 and $-3b$? of 4 and -5 ?

8. According to our study of the order of operations $a \times b$ gives the same product as $b \times a$. What, therefore, is the product of -13 and $+2$? of $-3b$ and $+5$?

The product of a positive number and a negative number in each of the above instances is negative. More briefly—*the product of any two numbers with unlike signs is negative.*

If by addition we seek to find the product of $2a^2 - ab - 5b^2$ by 3 we write the multiplicand three times:

$$\begin{array}{r} 2a^2 - ab - 5b^2 \\ 2a^2 - ab - 5b^2 \\ 2a^2 - ab - 5b^2 \\ \hline 6a^2 - 3ab - 15b^2 \end{array}$$

$$\therefore 3(2a^2 - ab - 5b^2) = 6a^2 - 3ab - 15b^2 \quad \text{Ans.}$$

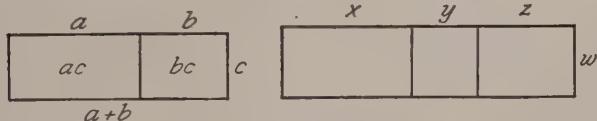
Notice that in the answer each term of $2a^2 - ab - 5b^2$ has been multiplied by 3:

$$3 \times 2a^2 = 6a^2$$

$$3 \times (-ab) = -3ab$$

$$3 \times (-5b^2) = -15b^2$$

When a polynomial is multiplied by a monomial, therefore, each term of the polynomial is multiplied in turn by the monomial. This may be illustrated by a diagram of a rectangle having the length $a + b$ and the width c . The area of the entire



rectangle is $c(a + b)$, and it is made up of two parts having the areas ac and bc .

$$\therefore c(a + b) = ac + bc$$

The second diagram represents the product $w(x + y + z)$. Give the area of each part, and fill in the blanks:

$$w(x + y + z) = \text{---} + \text{---} + \text{---}$$

EXERCISE 51

1. Using ruler and compasses, construct a rectangular diagram to illustrate the product of $a + b$ and a . (Let $a = 2''$, $b = 1\frac{1}{2}''$.) Find the total area of the diagram and that of each part.

2. Multiply $5c - 2d$ by 3, and prove by adding.

3. Multiply $2n^2 - 6n - 7$ by 4, and prove by adding.

Copy each of the following examples, and multiply:

$$\begin{array}{r} 4. \ 407 \\ \underline{3} \end{array}$$

$$\begin{array}{r} 5. \ 4h + 7 \\ \underline{3} \end{array}$$

$$\begin{array}{r} 6. \ 5 \text{ lb. } 3 \text{ oz.} \\ \underline{4} \end{array}$$

$$\begin{array}{r} 7. \ 6 \text{ ft. } 5 \text{ in.} \\ \underline{2} \end{array}$$

$$\begin{array}{r} 8. \ 6f + 5i \\ \underline{2} \end{array}$$

$$\begin{array}{r} 9. \ a + b \\ \underline{w} \end{array}$$

$$\begin{array}{r} 10. \ 3a^2 - 5x \\ \underline{4} \end{array}$$

$$\begin{array}{r} 11. \ a + b + c \\ \underline{x} \end{array}$$

$$\begin{array}{r} 12. \ 4a^2 + 7a - 6 \\ \underline{3} \end{array}$$

13. $4(3x^2 + 4x - 2)$. Check when $x = 1$.

Solution:

Check, when $x = 1$:

	Example:	Answer:
$\begin{array}{r} 3x^2 + 4x - 2 \\ \underline{4} \\ 12x^2 + 16x - 8 \end{array}$	$\begin{array}{r} 3 + 4 - 2 = 5 \\ \underline{4} \\ 4 \times 5 = 20 \end{array}$	$\begin{array}{r} 12 + 16 - 8 = \\ 20 \end{array}$

The remaining examples are to be checked:

14. Multiply $7a - 5$ by 4.

15. Multiply $5x^2 - 2x - 1$ by 7.

16. Multiply $10x - 7y$ by 6.

17. Multiply $5n + 2$ by -3 . Answer: $-15n - 6$

18. Multiply $6p + 4w$ by -5 .

19. Multiply $a + b$ by $-c$.

20. $4(9c - 7)$.

*Solution:**Check, when $c = 1$:*

$4(9c - 7) = 36c - 28$	Example:	Answer:
	$4(9c - 7) =$	$36c - 28$
	$4(9 - 7) =$	$36 - 28$
	$4(2) =$	
	$8 \longleftrightarrow 8$	

In a similar way, find the products indicated, and check:

21. $5(s^2 + 4s - 2)$

22. $x(x + y)$

23. $a(5a - 1)$

24. $2w(l - w)$ *Check, when $l = 5, w = 2$.*

25. $3(-4r^2 + 6r + 1)$

29. $-3R(7R + 2)$

26. $4v(3 - 2v)$

30. $-2q(p + q + 1)$

27. $10n(m - 4n + 1)$

31. $\frac{1}{2}h(a + b + c)$

28. $5a(x + y - 5z)$

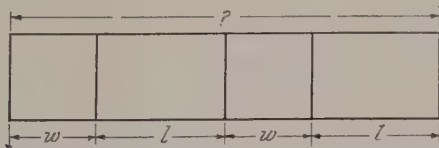
32. $ab(c + 3)$

33. A paymaster has to place in the pay envelope of each of fifteen laborers 3 five-dollar bills, 1 quarter, 2 dimes, and 3 pennies.

(a) What is the total number needed of each denomination?

(b) Find the total value of each denomination, and check the sum of their values with the total wages paid.

34. Multiply $3f + 1q + 2d + 3p$ by 15.

35. The four walls of a room h feet high may be developed into a rectangle:

What is the total length of the rectangle? What is the area?

36. Find the total area of the four walls of a room if $l = 14'$, $w = 10'$, $h = 9'$.

THE PRODUCT OF TWO NEGATIVE NUMBERS

In none of the preceding exercises did we have to find the product of two negative numbers such as -3 times -2 . To investigate what would be the product of -3 and -2 , let us try the example $-3(5a - 2)$:

$$\begin{array}{ccc} & (1) & (2) \\ & 5a - 2 & 5a - 2 \\ \text{The solution is either} & \frac{-3}{-15a - 6} & \text{or } \frac{-3}{-15a + 6} \end{array}$$

The checking of both solutions when $a = 1$ ought to tell which of the two is correct:

(1)	Answer:	(2)	Answer:
Example:		Example:	
$-3(5 - 2) =$	$-3(3) =$	$-3(5 \cdot 2) =$	$-3(3) =$
-9	$-15 - 6 = -21$	$-15 + 6 = -9$	$-15 + 6 = -9$
	$\longleftrightarrow$		$\longleftrightarrow$
	Not correct		Correct

It is evident that $-3(5a - 2)$ is $-15a + 6$ and that the product of -3 and -2 is $+6$.

Let us try another example; to multiply $6x - 4$ by -5 .

Solution: Check, when $x = 1$:

	Example:	Answer:
$6x - 4$	$6 - 4 = 2$	$-30x + 20 =$
-5	$-5 = -5$	$-30 + 20 =$
$-30x + 20$	$-5 \times 2 = -10$	-10
	$\longleftrightarrow$	

Because this solution checks correctly, we find that the product of -4 and -5 is $+20$.

We have just learned that the product of two negative numbers is a positive number. Since the product of two positive numbers is also a positive number we may say briefly that *the product of any two numbers with like signs is positive.*

EXERCISE 52

1. Multiply $7a - 3$ by -2 , and check when $a = 1$.
2. Multiply $16x - 5$ by -4 , and check when $x = 1$.
3. Multiply $6p - 3q$ by -3 , and check when $p = 1$, $q = 1$.

Copy each of the following, multiply, and check:

$$\begin{array}{r} 4. \quad 8x - 2 \\ \quad - 3 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 7a - 4 \\ \quad - 10 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 4m - x \\ \quad - 3x \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 2a - b \\ \quad - a \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 9x^2 - 5x - 1 \\ \quad - 2 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad x + y - z \\ \quad - 1 \\ \hline \end{array}$$

10. Multiply $lw + wh + lh$ by 2 .
11. Multiply $b + 2h$ by $2b$.
12. Multiply $v - 16t$ by t .
13. Multiply $3x - y$ by $-2x$. Check when $x = 2$, $y = 1$.

Find each of the following products, without checking:

$$14. (+7)(+8) \quad 19. (-2)(14) \quad 24. (-5x)(-5x)$$

$$15. (+7)(-9) \quad 20. (5x^2)(2) \quad 25. (-4)^2$$

$$16. (-5)(-11) \quad 21. (3a)(-4) \quad 26. (-ab)^2$$

$$17. (-8)(+12) \quad 22. (-5)(-4w) \quad 27. (-2rh)^2$$

$$18. 3(-15) \quad 23. (-6)(-6) \quad 28. (-abc)^2$$

$$29. 2cd(-3c)$$

$$30. 17(10 - 1)$$

$$31. 344(1000 + 2); n(1000 + a)$$

$$32. 432(1000 - 1); m(1000 - d)$$

$$33. 33(100 - 1); 27(99)$$

34. State a rule giving a "short cut" for multiplying any number by 99.

35. 268×999 . State a rule for multiplying a number by 999.

36. $-2m(6m + 7)$. Check when $m = 1$.

37. $-1(3r - 2)$. Check when $r = 1$.

38. $-(3s - 2t)$. Check when $s = 2, t = 1$.

39. $\frac{1}{2}n(a + 4l)$. Check when $a = 2, l = 1, n = 3$.

40. $\frac{1}{4}(12 \text{ ft. } 3 \text{ in.})$.

41. $.5(a - .5)$.

42. $.1(100h + 10t + u)$.

43. Using instruments, construct any rectangle having its length $2''$ more than the width.

44. If w represents the width of the rectangle in the preceding question what expression represents the area of the rectangle?

45. Find a formula for the area of a rectangle whose length is x inches more than its width. Ans. $A = w^2 + wx$.

46. Find a formula for the area (A) of a rectangle whose length is n times its width (w).

47. Draw a line $2\frac{1}{2}''$ long. At any point in the line construct, using instruments, a perpendicular $3\frac{3}{4}''$ long. Join the top of the perpendicular with the ends of the line, and find the area of the triangle formed.

48. Write a formula for the area of a triangle whose base is m times its altitude.

49. Give a formula for the area of a triangle whose altitude is d inches less than its base.

50. Add the product of a and $a + b$ to the product of b and $b - a$.

51. Show that $5(a + 2b) + 2(2a - b)$ is equal to $11a + 7b$.

52. Multiply, and add: $6(7 - 3x) + 7(4x - 6)$.

53. Multiply, etc.: $4(3x - y - 3) - 2(5x + 2y - 6)$.

54. Is $2l(l - w) - 2w(w - l)$ equal to $2(l^2 + w^2)$?

55. Multiply, and collect like terms

$$3a(a + 2b) - 5b(a + b) + 5(b^2 - a^2).$$

MULTIPLYING POWERS OF A QUANTITY

Although you are familiar with squares and cubes from your earlier work, occasionally in multiplication we meet with higher powers. With these we now need to get better acquainted. Let us study a few examples, some of them already familiar.

Example 1: Multiply R^2 by R

$$\begin{aligned} R^2 \text{ times } R \text{ means } R \cdot R \text{ times } R \\ = R \cdot R \cdot R \text{ or } R^3 \quad \text{Ans.} \end{aligned}$$

Example 2: Multiply $3a^2$ by $4a$

$$3a^2 \cdot 4a \text{ means } 3 \cdot a \cdot a \cdot 4 \cdot a$$

According to one of the principles of the order of operations the product of several numbers is not altered by changing the order of the numbers.

$$\begin{aligned} \therefore 3 \cdot a \cdot a \cdot 4 \cdot a \text{ is the same as } 3 \cdot 4 \cdot a \cdot a \cdot a, \text{ or } 12a^3 \\ \therefore 3a^2(4a) = 12a^3 \quad \text{Ans.} \end{aligned}$$

Example 3: Multiply $2b^2$ by $7b^3$

$$\begin{aligned} 2b^2 \cdot 7b^3 &= 2 \cdot b \cdot b \cdot 7 \cdot b \cdot b \cdot b \\ &= 2 \cdot 7 \cdot b \cdot b \cdot b \cdot b \cdot b \\ &= 14b^5 \end{aligned}$$

In practice it is not necessary to work the example in such great detail. There is a simple law for the multiplication of powers, which will become clear if you study the following:

- (a) $a^2 \times a = a \cdot a \times a = a^3$
- (b) $a^3 \times a^2 = a \cdot a \cdot a \times a \cdot a = a^5$
- (c) $a^2 \times a^2 = a \cdot a \times a \cdot a = a^4$
- (d) $a^4 \times a^2 = a \cdot a \cdot a \cdot a \times a \cdot a = a^{4+2} = a^6$

These examples show that *in multiplying powers of the same quantity we add the exponents* to find the exponent of that quantity in the product.

Example 4: Multiply $5r^2t^3$ by $6r^2t$

$$\begin{aligned} 5r^2t^3 \cdot 6r^2t &= 5 \cdot 6 \cdot r^2 \cdot r^2 \cdot t^3 \cdot t \\ &= 30r^4t^4 \end{aligned}$$

In practice we add the exponents mentally and write the answer at once, without rearranging, but particular care must be taken not to overlook any letter, coefficient, or exponent.

EXERCISE 53

1. Make a list of the first six powers of 2 from 2^1 to 2^6 .
2. What is 2^3 ? Show that $2^3 \times 2^3 = 2^6$.
3. Substitute 2 in place of a in parts (b) and (c) under Example 3 above, and check those examples.
4. Make a list of the first six powers of 10.
5. Multiply a^3 by a , and check when $a = 10$.
6. Multiply b^3 by b^3 , and check when $b = 10$.
7. What is the product of m and m^4 ? Check when $m = 2$.
8. Make a list of the first six powers of 1. Why is it unsatisfactory to check examples by substituting 1 when a mistake is liable to occur in the exponents?

Multiply:

- | | |
|------------------------|----------------------------|
| 9. $(3x^3)(2x^2)$ | 16. $\pi R(R + l)$ |
| 10. $(6ab^2)(3a^2b^2)$ | 17. $2\pi R(R + h)$ |
| 11. $2\pi R(\pi a^2)$ | 18. $lw(w - 4)$ |
| 12. $(-c^2d)(-c^2d)$ | 19. $-a(2a - bc)$ |
| 13. $a^2(a^3 - 2a^2)$ | 20. $3f^2n(-2f^3n)$ |
| 14. $x^3(x^2 + x + 1)$ | 21. $(-c^2d)(-c^2d)$ |
| 15. $ab(a + b)$ | 22. $(4pq^2r^3)(4pq^2r^3)$ |
23. Can you express Example 22 in any shorter way?
 24. What is the square of $3a^2x$?
 25. Multiply $t^2 + 2tu + u^2$ by t^2 .

What was the total number of runs scored by each team? How many runs were scored by the Giants in the first two innings? How much is 2 times zero? 9 times 0?

46. The number of automobiles sold by a dealer on six successive days was 4, 0, 2, 0, 0, 0. How many automobiles were sold in all? How many were sold the last three days?

47. What is the value of $0 + 0 + 0 + 0$? of $0 - 0$?

48. How much is 3×0 ? 0×17 ?

49. A man fails in business, having no money or resources whatever. How much will each of his twenty creditors receive from him?

50. What is $\frac{1}{20}$ of 0? $0 \div 20$?

If $a = 0$, $b = -2$, $c = 3$, and $d = -5$ find the value of:

51. $abcd$

59. $a + bd^3 + 6$

52. $b^2 + c^2$

60. $bc^2 (3c + d)$

53. $a + b + c + d$

61. $\frac{a}{3} + \frac{b}{2}$

54. $(b + c)^2$

62. $\frac{-4d}{b^2}$

55. $ab + bc + cd$

63. $c^2 - d^2$

56. $3b + \frac{c}{3}$

64. $(c - d)^2$

57. $bd \div c$

65. $\frac{7a - d^2}{c - b}$

58. $a^3 + b^3 + c^3$

66. $\frac{d^2 - 7a}{b - c}$

67. What number multiplied by $c - d$ would make the product $cx - dx$?

68. Write $am + ap$ as a product. Answer $a(? + ?)$.

69. What two numbers have the product $5b + 5c$?

70. Write $3x + 3a + 3r$ as the product of two numbers.

71. Write as a product $x^2 + 2xy$.

72. Of what two numbers is $a^2 - 7a$ the product?

73. $10x + 10y - 10$ is the product of what two numbers?
 74. Express $t - t^2$ in the form of a product.
 75. There are four main principles which guide us when multiplying, expressed by the four formulas below. Without turning back in your textbook, state each as a word rule.

- I. $(+a)(-b) = -ab$
 II. $a(b + c + d) = ab + ac + ad$
 III. $(-a)(-b) = +ab$
 IV. $a^m \times a^n = a^{m+n}$

MULTIPLICATION BY A BINOMIAL

The multiplication of two binomials may be studied by comparing it with the multiplication of two numbers in arithmetic:

$$\begin{array}{r} 21 \\ 32 \\ \hline 42 \\ 63 \\ \hline 672 \end{array} \qquad \begin{array}{r} 2t + 1 \\ 3t + 2 \\ \hline 4t + 2 \\ 6t^2 + 3t \\ \hline 6t^2 + 7t + 2 \end{array}$$

Check: Let $t = 10$, and one example checks the other.

First we multiplied each term of $2t + 1$ by 2 (just as first we multiplied 21 by 2 to get 42) and then by $3t$ (just as we multiplied 21 by 3 to get 63).

It is customary to multiply from the left as below; to multiply $2x + 5$ by $3x + 4$:

Solution:

Check, when $x = 1$:

$$\begin{array}{r} 2x + 5 \\ 3x + 4 \\ \hline 6x^2 + 15x \\ + 8x + 20 \\ \hline 6x^2 + 23x + 20 \end{array}$$

Example:

$$\begin{array}{r} 2 + 5 = 7 \\ 3 + 4 = 7 \\ 7 \times 7 = 49 \end{array}$$

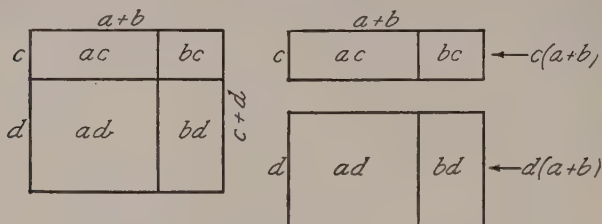
Answer:

$$\begin{array}{r} 6x^2 + 23x + 20 = \\ 6 \quad + 23 \quad + 20 = \end{array}$$

$$49 \longleftarrow \longrightarrow 49$$

Ans.

When any two binomials such as $a + b$ and $c + d$ are to be multiplied it can be shown by a diagram that the product is found by taking the product of each term of the multiplicand by each term of the multiplier in turn, and combining.



$$\begin{array}{r} a + b \\ c + d \\ \hline ac + bc \end{array} \quad \begin{array}{r} + ad + bd \\ \hline ac + bc + ad + bd \end{array}$$

The diagram has been split in two parts at the right, to show the product of $a + b$ by the first term c separate from that of $a + b$ by the second term d .

Example: Multiply $a - 2x$ by $4a - 3x$.

Solution:

Check, when $a = 2$, $x = 3$:

Example:

Answer:

$$\begin{array}{r|l} \begin{array}{r} a - 2x \\ 4a - 3x \\ \hline 4a^2 - 8ax \\ - 3ax + 6x^2 \\ \hline 4a^2 - 11ax + 6x^2 \end{array} & \begin{array}{r} 2 - 6 = -4 \\ 8 - 9 = -1 \\ -4 \times -1 = +4 \\ +4 \leftarrow \longrightarrow +4 \end{array} \end{array} \quad \begin{array}{l} 4a^2 - 11ax + 6x^2 = \\ 16 - 66 + 54 = \\ 70 - 66 = \end{array}$$

EXERCISE 54

1. Multiply $t + 5$ by $t + 2$. Check when $t = 10$.
2. Multiply $x + 7$ by $x + 3$. Check when $x = 3$.
3. Multiply $m + 1$ by $m + 4$. Check when $m = 2$.

4. Multiply $p + 8$ by $p - 2$. Check when $p = 3$.
5. Multiply $7x + 6$ by $5x - 3$. Check when $x = 2$.
6. Check Example 5 a second time by substituting $x = 3$.
7. Multiply $(l + 5)(l - 2)$. Check, let $l = 3$.
8. If in checking Example 7 you had let $l = 2$, what would have been the value of $l - 2$?

Multiply the following. Avoid a zero value for any quantity in checking; use 2 or 3 as the check numbers, except as you may be directed:

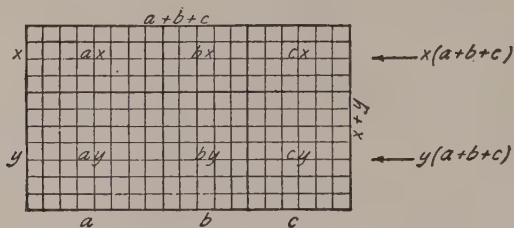
- | | |
|--------------------------|----------------------------|
| 9. $(7a + 3)(5a - 6)$ | 20. $(3s - 4t)(4s - 11t)$ |
| 10. $(n + 4)(n + 2)$ | 21. $(R^2 + 4)(R^2 + 6)$ |
| 11. $(w + 7)(w - 1)$ | 22. $(x^2 - 3y)(x^2 + 2y)$ |
| 12. $(h - 6)(h - 2)$ | 23. $(a - m^2)(a - 4m^2)$ |
| 13. $(3a + 5c)(11a + c)$ | 24. $(7ax + 3)(3ax + 7)$ |
| 14. $(7b + 2x)(3b - 2x)$ | 25. $(5b + 3c)(5c + 3b)$ |
| 15. $(4d - 3s)(10d - s)$ | [Hint: Rearrange.] |
| 16. $(2t + 3u)(3t + u)$ | 26. $(x^2 - 9)(9 - x^2)$ |
| 17. $(10x - a)(10x + a)$ | 27. $(x + a)(x + m)$. |
| 18. $(-x + 7)(-x + 4)$ | [Hint: The product |
| 19. $(-a - b)(-a - b)$ | has four terms.] |

28. Using instruments, construct a rectangular diagram like that which precedes this exercise, to illustrate the product in Example 27. Let $x = 2\frac{1}{2}"$, $a = 1"$, $m = 1\frac{3}{4}"$. Indicate the area of each part in terms of x , a , and m and check with your answer to No. 27.

29. Find the actual area in square inches of your diagram and of each part, using the given values of x , a , and m .

The multiplication of polynomials having three or more terms follows the same principles as do the preceding examples.

The rectangle of length $a + b + c$, width $x + y$, illustrates the fact that the product of any trinomial by a binomial is found by multiplying each term of the trinomial by each term of the binomial in turn and combining.



$$\begin{array}{r}
 a + b + c \\
 \underline{x + y} \\
 ax + bx + cx \\
 \qquad \qquad \qquad + ay + by + cy \\
 \hline
 ax + bx + cx + ay + by + cy
 \end{array}$$

Example: Multiply $2x^2 - 3xy - 4y^2$ by $3x - y$, and check:

Solution:

$$\begin{array}{r}
 2x^2 - 3xy - 4y^2 \\
 3x - y \\
 \hline
 6x^3 - 9x^2y - 12xy^2 \\
 \qquad - 2x^2y + 3xy^2 + 4y^3 \\
 \hline
 6x^3 - 11x^2y - 9xy^2 + 4y^3 \quad \text{Ans.}
 \end{array}$$

Check, when $x = 3, y = 2$:

Example:

Answer:

$$\begin{array}{r|l}
 18 - 18 - 16 = -16 & 6x^3 - 11x^2y - 9xy^2 + 4y^3 = \\
 9 - 2 = 7 & 162 - 198 - 108 + 32 = \\
 -16 \times 7 = -112 & 194 - 306 = \\
 -112 \longleftarrow & \longrightarrow -112
 \end{array}$$

EXERCISE 55

1. Prove by multiplying that the product of $4r^2 + 3r + 2$ and $3r + 1$ is $12r^3 + 13r^2 + 9r + 2$.

2. Multiply $2a^2 + 3a + 2$ by $3a + 1$. Check when $a = 2$.

3. Multiply $x^2 - 3x + 5$ by $x + 2$. Check when $x = 2$.

4. Show by multiplication that $(5x - 2a)(5x + 2a) = 25x^2 - 4a^2$.

5. What is the product of the two binomials $7a - 3b$ and $3a - 7b$? Check when $a = 3$, $b = 2$.

6. If the multiplicand is $5p - 3q$ and the multiplier is $4p + q$, find the product. Check when $p = 2$, $q = 3$.

7. Find by multiplication the square of $5r + 4s$. Check when $r = 2$, $s = 3$.

Multiply and check:

8. $(3n^2 - 2n + 1)(2n + 3)$

9. $(x^2 - 2x + 4)(x + 2)$

10. $(r^2 + 3r - 9)(r + 3)$

11. $(a^2 + ab + 5b^2)(a - 5b)$

12. $(2x + 3z)(4x^2 - 6xz + 9z^2)$

13. $(r - 1)(r + 1 + r^2)$. Rearrange in order of powers.

14. $(m - 3n)(-2mn - n^2 + 3m^2)$

15. $(x + a - b)(x - c)$. (The product has six terms.)

16. $(3x - 2y - 1)(2x + 5y)$

17. Multiply the product of $a + 5$ and $a + 4$ by $a + 2$.

Check when $a = 2$.

18. $(x + 3)(x - 2)(x - 3)$. Check when $x = 4$.

19. $(a + b)^2(a - b)$

20. $(t + 3)^3$

21. $(n^2 - 2n + 1)(n^2 + 2n + 1)$. Answer: $n^4 - 2n^2 + 1$.

22. $(a + b + 1)(a + b - 1)$. Answer: $a^2 + 2ab + b^2 - 1$.

23. $(a + b + c)^2$

24. The side of a square is s' . What is the area of a rectangle 2' wider and 5' longer than the square?

*25. Draw a rectangular diagram like that preceding this exercise, to illustrate the product of two trinomials $a + b + c$ and $x + y + z$. Represent the six letters by six convenient lengths.

*26. Draw a similar diagram to illustrate Example 23.

*27. Prove that if 1 foot is taken from the width of a square and 1 foot is added to its length, the resulting rectangle contains 1 square foot less than the original square.

MULTIPLICATION BY INSPECTION

Cross-products. When the first terms of two binomials are like terms, and the second terms also, it is not difficult to multiply the binomials mentally. Study this example and see how each term of the multiplication is obtained:

$$\begin{array}{r}
 3a + 2 \\
 \quad \times \quad 5 \\
 \hline
 4a + 5 \\
 12a^2 + 8a \\
 \quad + 15a + 10 \\
 \hline
 12a^2 + 23a + 10
 \end{array}$$

The $8a$ and the $15a$ are called cross-products and the sum of the two cross-products gives the middle term of the answer, $23a$. The first term $12a^2$ is the product of what two numbers? How was the last term obtained, $+10$?

In the first example below the cross-products will be $-12x$ and $-10x$. Can you get all three terms of the product mentally?

$$\begin{array}{r}
 5x - 3 \\
 \quad \times \quad 2 \\
 \hline
 4x - 2
 \end{array}$$

$$\begin{array}{r}
 3m + 5 \\
 \quad \times \quad 1 \\
 \hline
 2m + 1
 \end{array}$$

Try the second example mentally; what are the cross-products? What is the complete answer? The lines are drawn in each example to suggest how the cross-products are obtained.

EXERCISE 56

Multiply each of the following by inspection. If doubtful about any answer, multiply the example out on paper.

$$\begin{array}{r} 1. \ 3a + 4 \\ \quad 2a + 5 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \ 3t + 2 \\ \quad 7t - 4 \\ \hline \end{array}$$

$$\begin{array}{r} 2. \ 7r + 1 \\ \quad 5r + 1 \\ \hline \end{array}$$

$$\begin{array}{r} 7. \ 5x + 2y \\ \quad 3x + 2y \\ \hline \end{array}$$

$$\begin{array}{r} 3. \ 4b - 3 \\ \quad b - 2 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \ 4y - 7a \\ \quad 2y + 5a \\ \hline \end{array}$$

$$\begin{array}{r} 4. \ 6d - 1 \\ \quad 5d - 1 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \ ab - 4 \\ \quad ab - 3 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \ a + 5 \\ \quad a - 7 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \ q^2 - 2 \\ \quad q^2 + 1 \\ \hline \end{array}$$

Always try to see the cross-products mentally and multiply by inspection no matter how the multiplication is indicated. When the example is given in the form $(m + 4)(2m + 3)$ the cross-products may be shown by drawing a curved line to connect the terms forming each of the cross-products, thus:

$$\begin{array}{c} 8m \\ (m + 4) \overbrace{(2m + 3)} \\ 3m \end{array}$$

The middle term will be $8m + 3m$ or $11m$, and the complete product will be $2m^2 + 11m + 12$.

1. Multiply $c + 2$ by $3c + 2$.
12. Find the product of $8a + 5b$ and $8a + b$.
13. $(a + 3)(a + 7)$
21. $(3b - 2c)(3b - 2c)$
14. $(3y + 1)(4y + 5)$
22. $(5mn + 1)(5mn - 2)$
15. $(2n + 5s)(3n + 6s)$
23. $(10k - 3)(10k - 2)$
16. $(5v - 7t)(4v - 3t)$
24. $(3 + x)(2 + x)$
17. $(4r - x)(5r - x)$
25. $(4b + 1)(4b + 1)$
18. $(6p - 4q)(5p + 3q)$
26. $(2c - x)(2c + x)$
19. $(2n + 3)(2n + 3)$
27. $(a + 5)^2$
20. $(3x + 1)(3x - 1)$
28. $(a + d)(a - d)$

The Product of the Sum and the Difference of Any Two Numbers. In some of the examples in the preceding exercise the product has only two terms. Can you pick out any such examples? You will find in each case that the two binomials differ only in sign.

Try these mentally:

$$\begin{array}{r r r} a + 5 & 3x + 2 & s + m \\ \hline a - 5 & 3x - 2 & s - m \end{array}$$

How can you write the sum of a and b ? the difference of a and b ? the product of the sum and the difference? the sum of the squares of a and b ? the difference of the squares of a and b ?

Formula: $(a + b)(a - b) = a^2 - b^2$

EXERCISE 57

1. Copy, filling in the missing words: The product of the — and the — of two numbers is equal to the — of their —.

Multiply by inspection:

2. $(c + d)(c - d)$
4. $(z - h)(z + h)$
3. $(r + t)(r - t)$
5. $(x + 5)(x - 5)$

6. $(3y + 7)(3y - 7)$
 7. $(8v - 9s)(8v + 9s)$
 8. $(2R + r)(2R - r)$
 9. $(11m - 13n)(11m + 13n)$
 10. $(5 + 2x)(5 - 2x)$
 11. $(D - d)(D + d)$
 12. $(r_1 + r_2)(r_1 - r_2)$
 13. $(x^2 + 4)(x^2 - 4)$
 14. $(ah + 1)(ah - 1)$
 15. $(w' + w)(w' - w)$
 16. $(bcd - 3)(bcd + 3)$
 17. $(ae + c)(ae - c)$
 18. $(lr^2 + p)(lr^2 - p)$
 19. $(6a^2x + 10)(6a^2x - 10)$
 20. $(a + \frac{1}{2})(a - \frac{1}{2})$
 21. $(50 + 1)(50 - 1)$
 22. $(a + b) + x$
 $\underline{(a + b) - x}$
 $(a + b)^2 - x^2$ Ans.
 23. $(x + 5) + y$
 $\underline{(x + 5) - y}$
 24. $(p + q) + 3$
 $\underline{(p + q) - 3}$
 25. $[(a + b) + c][(a + b) - c]$
 26. $[(a - x) + 2b][(a - x) - 2b]$
 27. Multiply $(p - 2) + 10s$ by $(p - 2) - 10s$.
 28. Multiply $(n^2 + 2) + 2n$ by $(n^2 + 2) - 2n$. Simplify the product.
 29. What number multiplied by $l + w$ would give a product $l^2 - w^2$?
 30. What two numbers multiplied together would give $c^2 - b^2$?
 31. Write $y^2 - 16$ as the product of two binomials.

What two binomials would have as a product

32. $x^2 - a^2$

34. $a^2 - 25$

33. $m^2 - 9$

35. $l^2 - 4w^2$

The Square of a Binomial. Another special case of cross-products multiplication occurs when the two binomials are the same, *i.e.* when we are squaring a binomial.

Multiply by inspection:

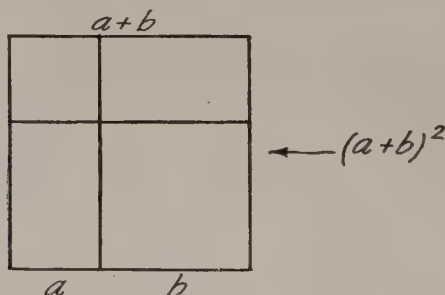
$$\begin{array}{r} a + 7 \\ a + 7 \\ \hline \end{array}$$

$$\begin{array}{r} x + 5 \\ x + 5 \\ \hline \end{array}$$

$$\begin{array}{r} c + d \\ c + d \\ \hline \end{array}$$

How does the middle term of the third example compare with the product of c and d ? Study how the middle term of each of the other products is obtained.

The square of $a + b$ is illustrated by the diagram of a square having the side $a + b$. Can you give the area of each of the four parts of the diagram?



Formula: $(a + b)^2 = a^2 + 2ab + b^2$

EXERCISE 58

1. Copy the following rule, supplying the missing words:
The square of the sum of two numbers is equal to the ——— of the first number, plus twice the ——— of the first number by the ———, plus the ——— of the ——— number.

2. Square $a + x$.

3. Square the sum of m and n .

4. What is the square of $3x + 4$?

Multiply by inspection:

5. $(x + y)^2$

7. $(n + 2)^2$

6. $(m + 3)^2$

8. $(t + u)^2$

9. $(2a + 5x)(2a + 5x)$

10. Square $x + (-4)$

$$\begin{aligned}\text{Solution: } x^2 + 2x(-4) + (-4)^2 \\ = x^2 - 8x + 16\end{aligned}$$

Notice that only the middle term is minus.

11. Square $a + (-b)$

12. Prove Examples 10 and 11 by multiplying in full.

13. $(a - r)^2$

23. $(t + 12)^2$

14. $(d - 5)^2$

24. $(n + \frac{1}{2})^2$

15. $(p - 8)^2$

25. $(x + x_1)^2$

16. $(2a - 7)^2$

26. $(1 - 9r^2)^2$

17. $(1 - 10w)^2$

27. $(D_1 + D_2)^2$

18. $(2m - 3d)^2$

28. $(30x + 1)^2$

19. $(xy + 6)^2$

29. $(30 + 1)^2$

20. $(3t - 4s)^2$

30. $(20 - 1)^2$

21. $(l^2 - 4w)^2$

31. 41^2

22. $(2mn + 3)(2mn + 3)$

32. 101^2

33. Write down any binomial (not taken from the book) and square it. Check by multiplying in full.

DIVISION BY A MONOMIAL

EXERCISE 59

1. What number multiplied by 3 will give the product 24?
2. What number multiplied by 3 will give the product -24?
3. Divide 24 by 3; divide -24 by 3.
4. What number multiplied by -4 gives -20?
5. Divide -20 by -4.
6. Divide -25 by -5.
7. What number multiplied by -2 gives +6?
8. Divide +60 by -20.
9. In Examples 4, 5 and 6 the two numbers have like signs. Is the quotient in each case positive or negative?
10. Consider carefully your answers to the preceding examples and then fill out this rule for signs in division:

When dividing two numbers that have like signs, the quotient is —; when dividing two numbers that have — signs, the quotient is —.

It is interesting to note the close relation between division and multiplication. Any question in division may be stated in the language of multiplication, just as any question in subtraction may be stated in the language of addition. (See page 95.)

11. What number multiplied by a^2 gives a product a^3 ?

12. Divide a^3 by a^2 . Divide a^3 by a .

13. What number multiplied by t^2 gives t^4 ?

14. Divide b^4 by b^2 .

15. Divide m^2 by m .

16. Divide q^4 by q . Divide x^4 by x^3 .

17. Fill out the rule for exponents in division: When dividing different — of the same quantity, — the exponent of the divisor from the — of the —, to get the — of that quantity in the quotient.

Supply the missing number in each example:

	DIVIDEND	DIVISOR	QUOTIENT
18.	$16r^2$	$2r$	—
19.	$20n^2$	$-4n$	—
20.	$15a^2b$	$3b$	—
21.	$-6m^3$	$-6m^3$	—
22.	πR^2h	h	—
23.	$-8D^3$	$+2D$	—
24.	—	$-5s$	$-2s$
25.	$12x^2y$	—	$3xy$
26.	$\frac{4}{3}\pi R^3$	$4\pi R^2$	—

27. What number multiplied by a gives the product $ab + ac$?

28. What is the quotient when $pb + pc$ is divided by p ?

29. Divide $bd + bt$ by b .

30. What number multiplied by x will give the product $mx - nx$?

31. Divide $cy - dy$ by y .

32. Divide $fp + fq + fr$ by f .

33. Find the quotient when $az - bz - cz$ is divided by z .

34. Divide 396 by 3; divide $3t^2 + 9t + 6$ by 3.

35. Divide 288 by 2; divide $2a^2 + 8a + 8$ by 2.

One should not use long division when the divisor is 12 or a smaller number. Divide each of the following by short division:

$$36. 1728 \div 12$$

$$44. 4 \overline{)16 \text{ lb. } 12 \text{ oz.}}$$

$$37. 5280 \div 3$$

$$45. 4 \overline{)16 p + 12 o}$$

$$38. 1760 \div 4$$

$$46. 2 \overline{)6 \text{ ft. } 7 \text{ in.}}$$

$$39. 63360 \div 12$$

$$47. m \overline{)am + an}$$

$$40. (43560 \div 8) \div 20$$

$$48. a \overline{)a^2 - 5a}$$

$$41. 3.1416 \div 4$$

$$49. c \overline{)cx^2 + cx - 10c}$$

$$42. 3 \overline{)9 \text{ pecks } 6 \text{ quarts}}$$

$$50. R \overline{)\pi R^2 + \pi Rh}$$

$$43. 3 \overline{)9p + 6q}$$

$$51. -3x \overline{)6x^2 - 12x}$$

Division may be indicated by words, by the division sign $\div$, or by a fraction. The numerator of the fraction is the dividend, and the denominator is the divisor. The three examples: $12x \div 3$, divide $12x$ by 3, and $\frac{12x}{3}$ all have the same meaning and the same result, $4x$.

Division examples may be proved by multiplication, or by numerical substitution.

Example: Divide $4a^3b + 6a^2b^2 + 4ab^3$ by $2ab$:

The quotient, by inspection, *Check*, by multiplication:
is $2a^2 + 3ab + 2b^2$

$$\begin{array}{r} 2a^2 + 3ab + 2b^2 \\ 2ab \\ \hline 4a^3b + 6a^2b^2 + 4ab^3 \end{array}$$

EXERCISE 60

Divide, and check by multiplication:

$$1. \frac{3mn + 12}{3}$$

$$5. \frac{6m^2 - 8m}{2m}$$

$$2. \frac{a^2 - 6a}{a}$$

$$6. \frac{p + prt}{p}$$

$$3. \frac{x^3 - 5x^2 - x}{x}$$

$$7. \frac{10t^2 - 15t}{-5t}$$

$$4. \frac{b - br}{b}$$

$$8. \frac{-3c^2d + 3cd^2}{-cd}$$

$$9. (\pi R^2 + \pi Rl) \div \pi R$$

$$10. (2x^2 - 4xy + 8xy^2) \div 2x$$

$$11. (n^4 - 4n^3 + n^2) \div (-n^2)$$

$$12. (3a^2b^2 - 6ab^3 + 9b^4) \div 3b^2$$

$$13. (-6r^2s + 12rs^2 - 8s^3) \div -2s$$

$$14. (\pi r_1^2h + \pi r_1r_2h + \pi r_2^2h) \div \pi h$$

15. By what number can you divide each of the terms of $5a + 5b$?

16. What divisor would be contained in each of the terms of $mx + my + mz$? (A "common divisor.")

17. Supply missing numbers in:

$$(a) (7c + 7d) \div \text{---} = \text{---} + \text{---}$$

$$(b) (ax + bx + cx) \div \text{---} = \text{---} + \text{---} + \text{---}$$

DIVISION BY A POLYNOMIAL

Compare these two division examples, step by step:

$$\begin{array}{r} 21 \\ 32 \overline{)672} \\ \underline{64} \\ 32 \\ \underline{32} \\ 0 \end{array}$$

$$\begin{array}{r} 2t + 1 \\ 3t + 2 \overline{)6t^2 + 7t + 2} \\ \underline{6t^2 + 4t} \\ + 3t + 2 \\ \underline{+ 3t + 2} \\ 0 \end{array}$$

The best check for division examples is the inverse operation, multiplication. You could have proved your multiplication examples by division, if you had then known how to divide.

EXERCISE 61

1. Divide 483 by 21. Place Examples 1 and 2 beside each other.

2. Divide $4t^2 + 8t + 3$ by $2t + 1$.

3. Divide $a^2 + 6a + 8$ by $a + 4$. Check by multiplication.

4. Divide $x^2 + 5x + 6$ by $x + 2$.

Divide and check:

5. $(m^2 + 6m + 9) \div (m + 3)$

6. $(21r^2 + 10r + 1) \div (3r + 1)$

7.
$$\frac{3x^2 + 5x + 2}{3x + 2}$$

8.
$$\frac{a^2 + 7ab + 12b^2}{a + 3b}$$

9.
$$\frac{7q^2 - 10q + 3}{q - 1}$$

10. $(12a^2 - 17a + 6) \div (4a - 3)$

11. $(12a^2 - 17a + 6) \div (3a - 2)$

Notice the divisors and quotients of the last two examples. Do they suggest another way to check division?

12. $(w^2 - 17wx + 60x^2) \div (w - 5x)$

13. $(a^2x^2 - 3ax - 18) \div (ax - 6)$

14. $(6v^2 + v - 2) \div (2v - 1)$

15. $(a^3 + 6a^2 + 12a + 8) \div (a + 2)$. Ans. $a^2 + 4a + 4$.

16. $(7x + 6 + x^3 + 4x^2) \div (x + 2)$. Rearrange the dividend.

17. $(n^3 + 27n - 27 - 9n^2) \div (n - 3)$

18. $(y^3 + 2y^2 - 16) \div (y - 2)$

19. $(x^3 + 27) \div (x + 3)$. Leave two vacant spaces between x^3 and 27 for the missing powers.

In many examples the divisor is not contained an integral number of times in the dividend and there is left a remainder.

20. Divide $a^2 + 2ab - b^2$ by $a - b$. Check by multiplying the divisor by the quotient and —— the remainder.

21. Divide $3z^2 - 7z + 5$ by $3z - 1$. Check.

22. Show by actual division that $a^3 + b^3$ is divisible by $a + b$.

23. Show that there is a remainder when $a^2 + b^2$ is divided by $a + b$.

24. Find the remainder when $x^3 - y^3$ is divided by $x + y$.
Ans. $-2y^3$.

25. What is the remainder when $a^3 + b^3$ is divided by $a - b$? Check.

26. Write any trinomial in x , and any binomial. Multiply the two, and prove the multiplication by division.

27. Cube $3a + 2$ by multiplication. Divide the result by $3a + 2$, and divide the quotient just obtained by $3a + 2$.

Use of Parentheses with Multiplication and Division. Parentheses like other parts of the language of algebra enable us to abbreviate and to simplify directions for examples.

Example: "From the product of 5 and the square of the sum of two numbers subtract twice the product of the sum and the difference of the two numbers."

Solution: Instead of this long statement of 28 words we simply write:

$$5(a + b)^2 - 2(a + b)(a - b)$$

In simplifying take the parentheses first:

$$= 5(a^2 + 2ab + b^2) - 2(a^2 - b^2)$$

Next do the multiplications:

$$= 5a^2 + 10ab + 5b^2 - 2a^2 + 2b^2.$$

Collect like terms:

$$= 3a^2 + 10ab + 7b^2 \quad \text{Ans.}$$

Check, when $a = 2$, $b = 1$:

Example:

Answer:

$5(3)^2 - 2(3) \quad (1)$ $= 5(9) - 2(3)$ $= 45 - 6$ $= 39$	$3a^2 + 10ab + b^2 =$ $12 + 20 + 7 =$
$\longleftarrow$	$\longrightarrow 39$

EXERCISE 62

1. Simplify $5(a + b) + 7(a - b)$. Check, when $a = 2$, $b = 1$.

Express with the aid of parentheses, then simplify:

2. To the sum of x and y add twice their difference.
3. From the difference of a and b subtract three times their sum.
4. From five times the larger of two numbers (a , b) subtract twice the sum of the two numbers.
5. From 9 times a certain number (n) subtract twice the sum of this number and 15.
6. Multiply five times the sum of h and $2D$ by the difference of $2D$ and h .
7. Multiply the square of $x + y$ by 3 and to the product add twice the square of $x - y$.
8. To the square of the sum of two numbers (a , b) add the square of the difference of the numbers.
9. From the square of the sum of x and y subtract the sum of the squares of x and y .
10. To the product of $a + x$ and $a - x$ add the product of $2x + a$ and $2x - a$.
11. From twice the square of $x - 5$ subtract the product of $x - 16$ and $x - 4$.
12. From the product of two numbers subtract the square of the difference of the numbers.

13. Simplify $5(x - 2a) + 10(x + a)$. Check $x = 1$, $a = 1$.

14. Simplify $(a + 7)(a + 3) + (a - 5)(a + 5)$. Check: $a = 2$.

15. Simplify $3(m + n)^2 - 2(m + n)(m - n)$.

16. When $A = 3m + 2$ and $B = m - 5$, what is the value of $3A + 2B$?

17. With the same values of A and B what is the value of the sum of A and B ? of the difference?

18. When $y = x + 5$ what is the value of $7y - 35$?

19. Substitute $3y - 2$ in place of x in the expression $2x + 3y + 4$.

20. Substitute $x + 3$ for y in the expression $xy + y^2 - 8$.

21. Simplify $(2x + 3)(3x - 2) - (6x + 5)(x - 1)$.

22. Simplify $(5c + d)(5c - d) + (3c - 2d)(c + 4d) - (7c + 1)(4c - 3)$.

23. Simplify by division, etc.:

$$\frac{15x + 10}{5} + \frac{x - 3x^2}{x}. \quad \text{Check when } x = 2.$$

24. Divide by inspection, and add:

$$\frac{a^2 + 2ab + b^2}{a + b} + \frac{a^2 - b^2}{a - b}$$

25. Divide and multiply as indicated:

$$\frac{x^2 + 10x + 16}{x + 2} \cdot (x - 8)$$

A Test or Review Lesson on the Four Fundamental Operations

Time, 40 Minutes

1. Express as a single term: $4m + m + 3m$.

2. Multiply $5a^2$ by $3a$.

3. $\frac{216}{9} = ?$

4. Add, and check when $n = 1$: $2n + 3$

$$2n + 1$$

$$\underline{2n - 1}$$

5. $3(4w - 2) = ?$

6. Yesterday the temperature at 8 A.M. was $+12^\circ$; today it was -5° . How much colder was it today than yesterday?

7. Multiply $4s + 3$ by $5s - 2$ and check when $s = 2$.

8. Find the sum of $3x - 4y + 7$ and $x + 4y - 5$.
Check when $x = 1$, $y = 1$.

9. $(r^2 - 2r + 5)(2r - 5) = ?$ Check when $r = 2$.

10. Collect like terms and add: $x^2 - 4 - x^2 + 4x - 4 - 5x + 10$. Check when $x = 1$.

11. Subtract $3p + 2q - 2r$ from $4p - q - r$. Check when $p = 1$, $q = 1$, $r = 1$.

12. Simplify $[(a + b) + (a - b)]$ and also $[(a + b) - (a - b)]$.

13. Divide $2\pi R^2 + 2\pi Rh$ by $2\pi R$.

14. Five persons finished a game with these scores: 30, 27, 6, 0, and -8 . What was the average of the five scores?

Omit checks in Examples 15 to 19:

15. $(n^3 - 6n^2 + 12n - 8) \div (n - 2) = ?$

16. Write in simplest form without parentheses $x^2 - (a - 2b)^2$.

17. By multiplication find $(3a - x)^3$.

18. Find the remainder, if any, when $c^2 + d^2$ is divided by $c + d$.

19. To twice the square of the sum of two numbers (a, b) , add the product of the sum and the difference of the numbers.

CHAPTER VII

EQUATIONS

You have already had considerable work with equations in the previous chapters. Refer back to this work, if necessary, in answering the following questions:

What is an equation?

What name is given to that part of an equation preceding the equality sign? that part following the equality sign?

What is meant by solving an equation?

What three axioms have been used in solving equations?

Tell what axiom you would use in solving each of the following equations:

(a) $x - 5 = 8$

(b) $x + 5 = 12$

(c) $3x = 9$

(d) $\frac{1}{2}x = 12$

How do you prove that your solution of an equation is correct?

When we solve an equation, we are finding a numerical value for the unknown quantity such that when this value is substituted for the unknown quantity the two members of the equation are equal numbers. This value is called the root of the equation and is said to satisfy the equation.

EXERCISE 63

1. Is 5 a root of the equation $2x + 9 = x + 14$? Why?
2. Is the equation $7x - 10 = 3x + 17$ satisfied when $x = 7$?
3. Is -8 a root of the equation $1 + 2n = n - 7$?

4. A class had to solve the equation $3(x + 8) = 4(13 - x)$. Some had 5 for the root of the equation, others, 4. Which were right?

5. Show that 2 is a root of the equation $9x - x^2 = 14$.

6. Is the equation $x^2 - 2x = 63$ satisfied when $x = 9$?

7. An algebra student claimed that 9 was the value of n in the equation $2n + 2(2n - 3) = 49$. Show, without solving the equation, whether he was right or wrong.

8. Write an equation containing four terms such that its root shall be 3.

Equations which contain only the first power of the unknown quantity are called *simple equations*. With the exception of the equations in Examples 5 and 6, all the equations in the preceding exercise are simple equations.

TRANSPOSITION

a. $x + 10 = 3x$

b. $x = 3x - 10$

If $x = 5$ is Equation **a** satisfied? Equation **b**?

If we examine these two equations we see that Equation **b** is obtained from Equation **a** by changing $+ 10$ from the left member of **a** to the right member, at the same time changing its sign from $+$ to $-$; we see, too, that this change did not disturb the equality of the two members, and did not change the value of x .

Consider carefully the following equations:

c. $2n = 60 - 3n$

d. $2n + 3n = 60$

How is Equation **d** obtained from Equation **c**?

If $n = 12$, what is the value of each member of Equation **c**?

If $n = 12$, are the two members of Equation **d** equal in value?

This changing of a number from one member of an equation to the other, changing its sign at the same time, is called transposition. Transposition shortens considerably the time required for the solution of an equation, as may be seen in the following solutions,—the first solution using the axioms; the second, transposition.

Example: Solve the equation $7x - 8 = 2 + 5x$.

First solution:

$$7x - 8 = 2 + 5x$$

Adding 8 to each member, $7x - 8 + 8 = 2 + 5x + 8$

Combining terms, $7x = 10 + 5x$

Subtracting $5x$ from each member,

$$7x - 5x = 10 + 5x - 5x$$

Combining terms, $2x = 10$

$$x = 5$$

Second solution:

$$7x - 8 = 2 + 5x$$

Transposing the -8

and the $5x$,

$$7x - 5x = 2 + 8$$

Combining terms, $2x = 10$

$$x = 5$$

Check:

Left member

Right member

$$7x - 8 =$$

$$2 + 5x =$$

$$35 - 8 =$$

$$2 + 25 =$$

$$27 \longleftarrow \hspace{1.5cm} \longrightarrow 27$$

Oral Exercise

Solve the following equations, using transposition:

1. $x - 5 = 7$

4. $y - 7 = 0$

2. $n - 6 = 1$

5. $m + 8 = 11$

3. $x - 10 = -3$

6. $g + 6 = -5$

7. $2x - 4 = 10$

8. $3t + 5 = 17$

9. $4 + x = 0$

10. $5 + n = -3$

11. $4m + 20 = 0$

12. $0 = 14 - x$

13. $5x = 2x + 18$

14. $6k = 21 - k$

15. $40 - 8N = 0$

16. $8 = 17 - 3R$

17. $5S = 24 + 3S$

18. $11 - z = z$

19. $3n - 5 = 2n$

20. $2r + 1 = r + 5$

EXERCISE 64

Solve each of the following and check your answer:

1. $4x + 11 = 31$

9. $72 = 6 + 11x$

2. $5y - 17 = 13$

10. $3x - 5 + 8x - 28 = 0$

3. $13x = 6x + 35$

11. $12y + 7 - 9y = -8$

4. $t = 18 - 3t$

12. $-4S = 7 - 5S - 3$

5. $5z - 9 = 2z$

13. $16 = 100 - 7d$

6. $3m + 14 = -4m$

14. $5 + 3x = x - 3$

7. $2k - 17 = k + 5$

15. $n = 17 - 4n + 3$

8. $3 + 2h = 11 - 6h$

16. $0 = 11x - 18 - 2x$

17. $3\frac{1}{2}x = 10 - 1\frac{1}{2}x$

If there are parentheses in the equations, they should first be removed. (Review pages 98-100.)

18. $2 - (7 - 3x) = (5x + 9) - 18$

Solution:

$$2 - (7 - 3x) = (5x + 9) - 18$$

Removing the parentheses, $2 - 7 + 3x = 5x + 9 - 18$

Transposing,

$$3x - 5x = 9 - 18 - 2 + 7$$

Combining terms,

$$2x = -4$$

$$x = 2$$

Check:

Left member

Right member

$$2 - (7 - 3x) =$$

$$(5x + 9) - 18$$

$$2 - (7 - 6) =$$

$$(10 + 9) - 18 =$$

$$2 - 1 =$$

$$19 - 18 =$$

$$1 \longleftarrow \hspace{15em} \longrightarrow 1$$

19. $3x + (5 - x) = 9$

20. $7 - (5m + 8) = 4$

21. $8 - (6 + 2n) = -8 + (4n - 2)$

22. $(3z + 6) - (4z + 3) = 5z - 21$

23. $(7k - 12) + 6 = 8 - (k - 6)$

24. $3x - (6x + 7) = 25 - (4x - 7)$

25. $(2y + 3) - (4y - 7) - (7y - 11) = 0$

26. $-(7r + 6) + 22 = 11 + (8 - 6r)$

27. $9 - (5m + 11) = 7m - (3 + 8m) - 15$

If there are any indicated multiplications, these should be performed before removing the parentheses.

28. $17 - 5(2 - 3x) + x^2 = 29 - (2 + x)(3 - x)$

Solution:

$$17 - 5(2 - 3x) + x^2 = 29 - (2 + x)(3 - x)$$

Performing the indicated multiplications,

$$17 - (10 - 15x) + x^2 = 29 - (6 + x - x^2)$$

Removing the parentheses,

$$17 - 10 + 15x + x^2 = 29 - 6 - x + x^2$$

Transposing, $15x + x^2 + x - x^2 = 29 - 6 - 17 + 10$

Combining terms, $16x = 16$

$$x = 1$$

Check:

Left member		Right member
$17 - 5(2 - 3x) + x^2 =$		$29 - (2 + x)(3 - x) =$
$17 - 5(2 - 3) + (1)^2 =$		$29 - (2 + 1)(3 - 1) =$
$17 - 5(-1) + 1 =$		$29 - (3)(2) =$
$17 + 5 + 1 =$		$29 - 6 =$
23	$\xleftarrow{\hspace{1.5cm}}$	23

29. $3(5y + 6) = 33$

30. $7 - 2(m + 6) = 3m$

31. $3(k - 2) - 5(k - 3) - 7(k - 4) = 1$

32. $3(2x + 8) - 10 = 52 - 5(7 - x)$
 33. $3 - 2(3z - 4) = 29$
 34. $5 + 3(2m - 6) = 2$
 35. $11 = 7 + 4(5 - 2m)$
 36. $3 - 10h = -6(2 + h)$
 37. $w(w + 5) = (w + 2)^2$
 38. $(2x + 3)(2x - 3) = 4x^2 + 3x$
 39. $(x + 6)(x - 5) = (x + 7)(x - 8)$
 40. $y^2 - (y + 3)(y + 5) = 1$
 41. $z(z + 7) - 2z(z - 6) = 19 - z^2$
 42. $(n - 5)^2 = n^2 + 5$
 43. $(v + 6)^2 - (v + 7)(v - 5) = 1$
 44. $(2x + 5)(x - 7) = (3x + 4)(x - 6) - (x^2 - 4)$
 45. $11 - (2k + 3)^2 = 4k(3 - k) - 46$
 46. $1.2x + 3.57 = .7x + 5.07$
 47. $4(0.2 + 0.5y) - 1.16 = 0.8y$
 48. $.5(z + .14) = 4 - 3(.21z + 16)$
 49. $3.14R = 2R + 5$. To three significant figures.

SYMBOLIC EXPRESSIONS

~ If you have any difficulty in expressing your answer to such a question as "By how much does a exceed b ?" make up a similar question involving only arithmetical numbers. If you ask yourself "By how much does 10 exceed 7?" you will see that the answer is $10 - 7$ and therefore that the answer to the original question is $a - b$. "How far will a ship go in h hours at the rate of r miles per hour?" Replace h by 3 and r by 15: "How far will a ship go in 3 hours at the rate of 15 miles per hour?" Since the 3 and the 15 are to be multiplied to obtain the distance sought, in just the same way the h and the r should be multiplied, and the answer to the original question is hr miles.

Oral Exercise

1. By how much does 12 exceed n ?
 2. By how much does s exceed p ?
 3. What is the difference between 23 and 18? between 14 and x ?
 4. From a ribbon 36" long a piece x inches long was cut off. How many inches of ribbon were left?
 5. If there are n pupils in your algebra class and b of them are boys, how many are girls?
 6. If one part of 15 is 8, what is the other part?
 7. If one part of 15 is n , what is the other part?
 8. C is separated into two parts; if one part is 10, what is the other?
 9. The sum of two numbers is 18. If one of the numbers is 6, what is the other number?
 10. Two numbers added together make 20. One number is n ; what is the other?
 11. One number is 6 more than another. If the smaller number is n , what is the larger?
 12. If in Example 11, the larger number were x what would be the smaller?
 13. One number is three times as large as another. If the smaller is y , what is the larger? If the larger is y , what is the smaller?
 14. How many ounces are there in 2 pounds? in p pounds?
 15. How many yards are there in 60 feet? Express as a formula the number of yards (y) in f feet.
- Consecutive numbers are whole numbers arranged in the order of ordinary counting, such as 6, 7, 8, 9.
16. What is the difference between any two consecutive numbers?
 17. What number is 1 more than 7? 1 less than 7?
 18. What number is 1 more than -5 ? 1 less than -3 ?

19. What number is 1 more than n ? 1 less than n ?
20. What are the seven consecutive numbers from -3 to $+3$ inclusive?
21. If n represents the smallest of three consecutive numbers, what are the other two?
22. Give the five consecutive numbers of which 7 is the largest.
23. Write three consecutive numbers of which n is the largest.
24. What is the difference between any two consecutive odd numbers?
25. Give four consecutive odd numbers of which m is the largest; of which m is the smallest.
26. If n is an odd number, is $n + 1$ odd or even?
27. If m is an even number, what are the next two consecutive even numbers?
28. If n is an even number, is $n - 1$ odd or even? $n - 2$? $n - 4$?
29. Write three consecutive numbers of which the smallest is $2n$.
30. Is $2n$ an odd number or an even number? $2n + 1$?
31. What is the product if the multiplicand is 5 and the multiplier a ?
32. What is the quotient when the dividend is a and the divisor 3? When the dividend is 3 and the divisor a ?
33. What number divided by 10 will give x as a quotient?
34. Find the dividend if the divisor is 7, the quotient 8, and the remainder 3.
35. Find the dividend, if the quotient is n , the divisor 6, and the remainder 2.
36. Write a formula for the dividend (D) in terms of the divisor (d), the quotient (q), and the remainder (r).
37. What is the cost of n loaves of bread at 12 cents each?

38. What is the cost of one loaf of bread if 6 loaves cost c cents?

39. A dealer paid d dollars for a dozen hammers. What represents the cost of one hammer?

40. If a man walks 3 miles per hour how far will he walk in h hours?

41. At what rate is a man walking if he walks m miles in 5 hours? 5 miles in h hours?

42. Give a formula for the distance (d) traveled by a man in t hours at the rate of r miles per hour.

43. Give a formula for finding the rate when the distance and time are known. (r, d, t .)

44. A is x years old and B is 15. How much older is A than B?

45. How old will C be in 6 years if he is y years old now? How old was he 6 years ago?

46. A has \$15 and B has \$10. How much will each have if B gives A d dollars? if A gives B d dollars?

47. How many cents are there in n nickels? in d dimes?

48. I have q quarters, d dimes, and n nickels. How many cents have I?

49. Give a number which is 12 more than x ; 12 less than x .

50. The width of a rectangular field is w feet. Its length is 10' more than its width. What represents its length? its perimeter? its area?

51. If the side of a square is x inches long, what is the perimeter of the square? the area?

52. If the perimeter of a square is x inches, how many inches in the side of the square? how many square inches in the area of the square?

53. The side of a square is S inches in length. What are the dimensions of a rectangle which is 3" narrower and 5" longer?

54. From a stick $36''$ long 5 pieces each $3''$ long were cut off. What is the length of the piece left?

55. From a stick l'' long, p pieces each i'' long are cut off. How many inches are there in the piece left?

56. Express 6% as a decimal.

57. Express $p\%$ as a common fraction.

58. What is 5% of a ? $a\%$ of 5?

59. A man purchased an article for d dollars and later sold it at a gain of 10% . What is the gain in dollars? the price at which he sold it?

60. If a chair which cost d dollars was sold at a loss of $r\%$, what was the loss in dollars? the selling price?

Express each of the following statements as an equation:

61. Twice a certain number is 14.

62. The sum of 23 and x is 42.

63. n exceeds 10 by 7.

64. The double of x exceeds the half of x by 3.

65. The sum of two numbers of which one is three times the other is 36.

66. The sum of two numbers which differ by 3 is 36.

67. Five per cent of n dollars is twenty dollars.

68. $r\%$ of \$150 is \$45.

69. It takes a train 4 hours to travel 100 miles at the rate of x miles per hour.

70. Write a formula for the average daily receipts (D) of a trolley company if the receipts for a week of 7 days average w dollars.

71. Write a formula for the amount of a man's bank balance (B) if his deposits have totaled d dollars and his withdrawals w dollars.

72. What would the previous formula become if in addition his account has been credited with i dollars interest?

PROBLEMS TO BE SOLVED BY EQUATIONS

The word problems in Chapter V which you solved by the help of equations were such that you could translate the words of the problems directly into the symbols of algebra, forming an equation. In the problems of this chapter there are usually two numbers involved, and the words of the problem embody two statements. After choosing a suitable letter to represent one of the unknown numbers, one of the statements is used to obtain an expression which shall represent the other unknown number, and then the second statement is used in making the equation.

Example 1: 100 is separated into two parts so that one part is 7 times the other. Find the two parts.

First solution:

The two statements are:

(1) 100 is separated into two parts

(2) One part is 7 times the other

First let us choose x to represent the smaller part. Then, in order to obtain an expression for the second part, we make use of statement (1). $100 - x$ represents the second or larger part. Now from statement (2) we form our equation. One part is 7 times the other.

$$100 - x = 7x$$

$$100 - x = 7x$$

$$-x - 7x = -100$$

$$-8x = -100$$

$$x = 12\frac{1}{2}, \text{ the smaller part}$$

and $100 - x = 87\frac{1}{2}, \text{ the larger part.}$

Second solution:

If we should choose statement (2) as the one from which to obtain an expression for the larger part, the plan of our solution would be:

Let x = the smaller part
 Then $7x$ = the larger part.

Now statement (1) must be used for the equation.

100 is separated into two parts

$$100 = x + 7x$$

$$-x - 7x = -100$$

$$-8x = -100$$

$$x = 12\frac{1}{2}, \text{ the smaller part}$$

$$7x = 87\frac{1}{2}, \text{ the larger part.}$$

Example 2: Mr. Holmes is three times as old as his son, and the difference of their ages is 32 years. What are their ages?

First solution:

The two statements are:

(1) Mr. Holmes is three times as old as his son

(2) The difference of their ages is 32 years.

Let x = the age of the son

Then, using statement (1)

$3x$ = the age of Mr. Holmes

(2) The difference of their ages is 32 years

$$3x - x = 32$$

$$2x = 32$$

$$x = 16, \text{ the son's age}$$

$$3x = 48, \text{ Mr. Holmes' age}$$

Second solution:

Let x = the age of the son

Using the statement (2)

Then $x + 32$ = the age of Mr. Holmes.

- (1) Mr. Holmes is three times as old as his son.

$$x + 32 = 3x$$

$$x - 3x = -32$$

$$-2x = -32$$

$$x = 16, \text{ the son's age.}$$

$$x + 32 = 48, \text{ Mr. Holmes' age.}$$

EXERCISE 65

1. Find two numbers such that (1) the difference of the two numbers is 48, and (2) one of the numbers is 5 times the other. (If possible give two solutions as in Examples 1 and 2 above.)

2. Find two numbers such that (1) the first number is double the second, and (2) the sum of the two numbers is 81.

3. (1) The sum of two numbers is 100, and (2) one of the numbers is 12 more than the other. Find the numbers.

4. The sum of two numbers is 57 and their difference is 13. What are the numbers?

5. Separate 125 into two parts, one of which is 5 more than the other.

6. The larger of two numbers is 10 more than the smaller. Their sum is 64. Find the two numbers.

7. The sum of two numbers is 89. One number is 15 less than the other. What are the two numbers?

8. The sum of two consecutive numbers is 45. What are the numbers?

The second statement about the two numbers is here implied in the word "consecutive." The full statements are

(1) Two numbers are consecutive. (2) Their sum is 45.

9. Find two consecutive numbers whose sum is 143.

10. Find two consecutive numbers such that the sum of the first and twice the second is 38.

11. The sum of two consecutive even numbers is 34. Find them.

12. The length of a rectangle is 12' more than the width. Find the length and the width if the perimeter is 44'.

13. Joe is three years older than Bill. Five years from now the sum of their ages will be 27 years. Find their ages now.

When there are more than two numbers involved in a problem, there must be an additional statement, either expressed or implied, for each additional number. A problem which involves three numbers must contain three distinct statements either directly expressed, or implied in some word or group of words.

Example: The perimeter of a triangle is 38". The first side is double the second and the third side is 6" longer than the second. What are the lengths of the sides?

Solution:

There are three numbers involved — the lengths of the three sides. The three statements are:

- (a) The perimeter of a triangle is 38".
- (b) The first side is double the second.
- (c) The third side is 6" longer than the second.

Let x = the length of the second side.

Then from statement, (b) $2x$ = the length of the first side and from statement (c), $x + 6$ = the length of the third side.

Now using statement (a) we make the equation

$$2x + x + x + 6 = 38$$

$$2x + x + x = 38 - 6$$

$$4x = 32$$

$$x = 8, \text{ the length of the second side}$$

$$2x = 16, \text{ the length of the first side}$$

$$x + 6 = 14, \text{ the length of the third side}$$

Check:

$$(a) \quad 8 + 16 + 14 = 38.$$

$$(b) \quad 16 \text{ is twice } 8.$$

$$(c) \quad 14 \text{ is } 6 \text{ more than } 8.$$

EXERCISE 66

1. Find three numbers such that the sum of the numbers is 29, the second number is double the first, and the third number is 5 more than the first.

2. How can \$70 be divided among A, B, and C so that A will get \$10 more than B, and C will get three times as much as B?

3. A has \$17 more than B and \$18 more than C; together they have \$85. How many dollars does each one have?

4. The perimeter of an isosceles triangle is 42", and the base is 3" longer than either arm. Find the three sides of the triangle.

5. Find three consecutive numbers whose sum is 48.

6. Find three consecutive numbers such that the sum of the first, twice the second, and three times the third is 38.

7. Find three consecutive odd numbers such that their sum is 105.

8. The sum of four consecutive numbers is 90. What are they?

9. The sum of three numbers is 336. The second number is 13 times the first and the third is equal to the sum of the other two. What are the three numbers?

10. Find the three sides of a triangle whose perimeter is 83", if the second side is 10" longer than the first and the third side is 12" longer than the second.

Problems Based on the Product Relation. The product relation forms the basis for a great many problems in algebra. Each of these problems contains three different quantities so related that the product of two of them, with certain restrictions, is equal to the third. Examples of such quantities are:

- (1) length, width, and area of a rectangle
- (2) rate, time, and distance

- (3) cost of one article, number of articles, and total cost
 (4) number of coins, value of a single coin, and the total value
 (5) principal, rate, and interest.

It is convenient in the solution of this type of problem to arrange the work in three columns as illustrated in the following problems:

Example 1: To build a shack in the woods, eight boys each agreed to contribute a certain amount. Two boys moved out of town and the others had to pay \$5 apiece additional. What was the cost of the shack?

Solution:

Let x = the number of dollars that each boy had agreed to contribute.

Now observe that the product relation is involved for *the share of one boy* multiplied by *the number of boys* will give *the cost of the shack*.

Arrange the work in three columns:

	Share	Number of Boys	Cost of Shack
First plan	x	8	$8x$
Second plan	$x + 5$	6	$6(x + 5)$

Now since the cost of the shack is the same under either plan,

$$8x = 6(x + 5)$$

$$8x = 6x + 30$$

$$8x - 6x = 30$$

$$2x = 30$$

$x = 15$, the number of dollars each boy had agreed to pay.

$8x = 120$, the cost of the shack.

Check:

Eight boys, each paying \$15, would raise \$120.

Six boys, each paying \$20, would raise \$120.

Example 2: A man invested a certain sum of money at four per cent, and twice as much at five per cent. If his total annual income from the two investments was \$245, how much had he invested at each rate?

Solution:

Let x = the number of dollars invested at 4%.

	Investment	Rate	Interest
First sum.....	x	.04	.04 x
Second sum.....	$2x$	.05	.05($2x$)

Since his income will be the sum of these two amounts of interest,

$$.04x + .05(2x) = 245$$

$$.04x + .10x = 245$$

$$.14x = 245$$

$$x = 1750, \text{ the first investment}$$

$$2x = 3500, \text{ the second investment}$$

Check: (Let each pupil check these results.)

EXERCISE 67

1. Two rectangles have the same length, 24", but the width of one is 5" more than the width of the other. The sum of the areas of the two rectangles is 600 square inches. Find the width of each rectangle.

2. The length of a certain rectangle is 8' more and the width 6' less than the side of a certain square. If the area of the rectangle is 16 square feet less than the area of the square, what is the side of the square?

3. A freight train makes the run between two towns in 18 hours; an express whose rate of speed is 25 miles per hour more than the rate of the freight takes 8 hours between the same towns. What is the speed of each train?

4. While waiting for a train, a man has four hours at his disposal. He walks at the rate of three miles per hour to a

nearby village and drives back at the rate of 5 miles per hour, arriving just in time to catch the train. How far away was the village?

5. A man spent a certain sum for apples at 4¢ each. If they had cost 3¢ each, he would have had 8 apples more for the same money. How many apples did he buy?

6. Ten boys, on a picnic, agree to share the expenses equally. As two boys found themselves unable to pay their shares, each of the others had to pay 30 cents more. What was the cost of the picnic?

7. I have in my pocket ten coins, dimes and nickels. If the value of the ten coins is 65 cents, how many dimes are there?

8. The cost of 15 stamps was 38 cents. Some were two-cent stamps; the others, three-cent stamps. How many of each kind were there?

9. The parcel post rates to the third zone are "six cents for the first pound and two cents for each additional pound." If it cost 22 cents for a certain parcel to be sent to the third zone, how many pounds did it weigh?

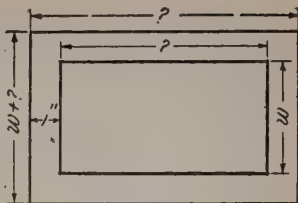
10. A man invested \$15,000, part of it at 5%, the rest at 6%. The income from the six per cent investment was \$185 more than that from the five per cent investment. How much did he invest at each per cent?

11. A man invested \$1200 at a certain per cent and \$3700 at a rate greater by 1 per cent. If his total annual income was \$282, what rate per cent did he get on each investment?

12. The length of a rectangle is 3' more than twice the width. The perimeter of the rectangle is 42'. Find the length and the width of the rectangle.

13. A baseball team purchased 8 bats and a dozen balls for \$36. If a ball cost 50 cents more than a bat, what is the cost of each?

14. A picture is 3" longer than it is wide. It is surrounded by a frame 1" wide. If the outside perimeter is 34", what are the dimensions of the picture?



15. It took a man 8 hours to travel from one town to another.

Part of the time he walked at the rate of 4 miles per hour; the rest of the time he was driven in an automobile at an average rate of 24 miles per hour. If the towns were 92 miles apart, how many hours did he walk?

16. Separate 240 into two parts so that if three times the larger part is diminished by 39 the result will be the same as when 46 is added to twice the smaller part.

17. The length of a rectangle is 4' more than its width. A second rectangle is 4' longer and 2' narrower than the first rectangle but has the same area. What are the dimensions of the rectangle?

18. A farmer spent \$480 in buying cows and pigs. He bought 25 more pigs than cows. If the cows cost him \$90 apiece and the pigs \$7.50 each, how many pigs did he buy?

19. A boy is four years older than his sister. In 5 years the sum of their ages will be 28 years. How old is each now?

20. A grocery store sells two grades of coffee, one at 25 cents a pound, the other at 35 cents a pound. A manager of a restaurant bought \$23.75 worth of coffee at this store. If he bought twice as many pounds of the more expensive kind as of the cheaper, how many pounds of each kind did he buy?

21. A square and a rectangle have the same area. The width of the rectangle is 3' less and the length is 4' more than the side of the square. What is the side of the square?

22. A carpenter was engaged for 30 days on the condition that for every day he worked he was to receive \$10.50 and for every day he was not at work he was to forfeit \$5. If

he received \$253 at the end of the contract period, how many days did he work?

A TEST ON EQUATIONS

Time, 40 Minutes

Solve each equation, and check.

1. $6n - n = 300$
2. $3x - 7 = 14 - 4x$
3. $(L + 4)(L - 1) = (L + 1)^2$
4. $2\frac{1}{2}x - 15 = 0$
5. $3(m + 1) = 4(m - 5) + 16$
6. $y + 5(2y + 1) = 3(3y - 2) + 5$
7. $3\frac{1}{7}D = 12$
8. $10 - (3n - 5) = 9 - (7n + 2)$
9. $1.06p = 35$ (to the nearest hundredth)
10. From the formula $T = 4bh + 2b^2$ find h if $T = 78$ and $b = 3$.
11. $w(3w) - (w - 1)(3w + 1) = 13$

(The above questions have been carefully graded as to difficulty. If each correct solution is counted as $\frac{2}{3}$ and each check as $\frac{1}{3}$, the median score of a typical class should probably be not less than 6.)

A TEST ON FORMULAS AND PROBLEMS

PART ONE — MAKING FORMULAS

1. What would be the cost of n apples at 4 cents apiece?
2. A radio set which cost c dollars was sold at a profit of \$10. For how many dollars did it sell?
3. A rectangle is twice as long as it is wide. Its width is w inches. What is its length? its perimeter?
4. The volume of a pyramid is equal to one-third the product of its base and altitude. Find the volume if the base is 100 and the altitude 12.

5. Write a formula for the volume of a pyramid (V , B , H).

6. For an advertisement one must pay twenty cents, plus an additional amount equal to two cents for each word. What would be the cost of an advertisement of fifteen words?

7. Express as a formula the cost of an advertisement of n words.

8. Show that the difference between the square of the sum of two numbers (a , b) and the square of the difference of the numbers is always four times the product of the numbers. Verify this statement when the two numbers are 7 and 3.

PART TWO — SOLVING PROBLEMS

Solve each of these problems by the use of an equation:

1. Eight times a certain number is equal to 200. What is the number?

2. If 4 is subtracted from three times a number the result is 23. What is the number?

3. The larger of two numbers is 6 more than the smaller. The sum of the two numbers is 72. Find the numbers.

4. Divide 20 into two parts one of which increased by 3 shall be equal to the other diminished by 9.

5. Find the side of a square field such that if the field were 3 feet longer and 2 feet wider the area would be 46 square feet more.

CHAPTER VIII

FACTORING

If we multiply 7 by 11 the product is 77, and the two numbers 7 and 11 are called the factors of 77.

The factors of a quantity are the numbers whose product is the quantity.

A quantity which has no factors other than itself and 1 is said to be prime.

EXERCISE 68

1. What are the factors of each of the following numbers: 14, 33, 26, 35, 91?

2. One factor of 153 is 9. What is the other factor? How is the other factor obtained?

3. What is the other factor of 60 if one factor is 2? 3? 6? 15?

4. Factor 24 in three different ways. (Two factors in each set)

5. Write four sets of two factors for 40.

6. How many sets of two factors can you find for 72? Give them.

7. Is there any number which is a factor of both 14 and 21? What is it?

8. What factor is common to the two numbers 15 and 35?

9. Find a common factor of 39, 15 and 33.

10. There are four prime numbers between 10 and 20. What are they?

11. How many prime numbers are there between 20 and 30? What are they?

12. Find the prime factors of each of the following (three factors for each): 30, 42, 66, 105, 28, 1001.

13. What prime factor is common to all even numbers? to all numbers that end in 5? What two prime factors are common to all numbers that end in 0?

14. If the product of two numbers is $3pq$ and one of the numbers is $3q$, what is the other number?

15. What is the other factor of $10a^2b^2$ if one factor is a^2 ? $5ab$? $2a^2b$? $10ab^2$?

16. Name two prime factors which are common to 30 and 18; to abc and acx .

17. The product of a and $x + y$ is $ax + ay$. What are the factors of $ax + ay$?

18. One factor of $12m^2a + 8mb$ is $4m$; how can the other factor be obtained? What is the other factor?

19. What factor appears in each term of $at + ar + av^2$? Write $at + ar + av^2$ as the product of two factors.

20. What factor is common to all the terms of $3x + 3y + 15$? Factor $3x + 3y + 15$.

COMMON FACTORS

Typical Example: $af + bf + cf = f(a + b + c)$

This method of factoring is useful whenever there is a factor common to all the terms of an algebraic expression.

Oral Exercise

Find the factors of each of the following:

1. $5a + 15$

7. $ab + a$

2. $2a + 6$

8. $ax - 5x$

3. $3x + 3y$

9. $\pi D + \pi d$

4. $pq + pr$

10. $7a + 21b$

5. $2a - 6$

11. $x^2 - 2x$

6. $4a - 8b$

12. $7 - 7d$

13. $nd - d$

17. $az + bz - cz$

14. $t^2 + 2t$

18. $2x + 4y \div 6z$

15. $3y^2 + 6y$

19. $\frac{1}{2}r + \frac{1}{2}s$

16. $ax + mx + nx$

20. $\frac{1}{3}a - \frac{1}{3}x$

21. What prime factors are common to $12a^2b$ and $18ab^2$?
What is the product of these prime factors?

The product of all the prime factors which are common to several numbers is the *highest common factor* of those numbers. Thus, 2 is a common factor of $10x^2$ and $6x^3y$, but the highest common factor is $2x^2$.

22. What is the highest common factor of 20 and 50? of 12 and 30?

23. What is the highest common factor of abc and acx ? of $18xy^2$ and $24x^3y$? of $6m^2n^3p$, $9mn^2p^2$, and $30m^3n^3p^3$?

24. What is the highest common factor of the two terms of $14a^2b^3 + 21ab^2$?

Example: Factor the expression $18m^2n + 30mn^2$ and prove your work to be correct.

Solution: The highest common factor of the terms $18m^2n$ and $30mn^2$ is $6mn$; therefore,

$$18m^2n + 30mn^2 = 6mn(3m + 5n)$$

Check: The work may be checked by multiplying the factors together:

$$\begin{array}{r} 3m + 5n \\ 6mn \\ \hline 18m^2n + 30mn^2 \end{array}$$

EXERCISE 69

Factor the following and check your work by multiplication. In each example, first decide what is the highest common factor of its terms.

1. $7x^2 + 14xy$

3. $2abc - 5abx$

2. $3a^2 - 21b^2$

4. $3mn - 15m^2y$

- | | |
|--------------------------------------|---|
| 5. $vt + 8t^2$ | 12. $7mn + 21nx + 35ny$ |
| 6. $2\pi RH + 2\pi R^2$ | 13. $11x - 55y + 121xy$ |
| 7. $21x^2y + 28xy^2$ | 14. $\pi R^2 + \pi Rr + \pi r^2$ |
| 8. $2an + dn^2 - dn$ | 15. $3n^4 + 12n^3 - 18n^2$ |
| 9. $22x^2 + 7x$ | 16. $7x - 36xy - 21xz$ |
| 10. $2x^3 - 4x^2 + 8x$ | 17. $\frac{1}{2}bh + \frac{1}{2}ch - \frac{1}{2}dh$ |
| 11. $5a^2bc^2 - 15a^2b^2c$ | 18. $15a^2x^2 - 30ax^3 + 25a^3x$ |
| 19. $12a^4x^4 - 18a^2x^2 + 36a^3x^2$ | |
| 20. $-4a^3b + 6a^2b^2 - 4ab^3$ | |

Rewrite each of the following formulas after factoring the right member:

- | | |
|---|-----------------------------|
| 21. $P = 2l + 2w$ | 24. $l = \pi d_1 - \pi d_2$ |
| 22. $S = b - br$ | 25. $a = p + prt$ |
| 23. $l = \pi D + \pi d$ | 26. $S = vt + 16t^2$ |
| 27. $S = \frac{an + ln}{2} = \frac{n}{2} (?)$ | |

Divide your paper vertically into halves and in the left half find the value of each of the following expressions as given. In the right half, first factor the expression and then find its value, checking the two results.

$$R = 14, r = 7, D = 28, d = 14, l = 10, H = 6, \pi = 3\frac{1}{7}$$

$$28. \pi R^2 + \pi r^2$$

Solution:

I	II
$\pi R^2 + \pi r^2 =$	$\pi(R^2 + r^2) =$
$3\frac{1}{7}(14)^2 + 3\frac{1}{7}(7)^2 =$	$3\frac{1}{7}(14^2 + 7^2) =$
$\frac{22}{7} \cdot \cancel{196}^{28} + \frac{22}{7} \cdot \cancel{49}^7 =$	$\frac{22}{7} (196 + 49) =$
$616 + 154 =$	$\frac{22}{7} \cdot \cancel{245}^{35} =$
$770 \longleftarrow \hspace{10em} \longrightarrow 770$	

$$29. \pi r l + \pi r^2$$

$$30. \pi R^2 H - \pi r^2 H$$

$$31. \frac{1}{4} \pi D^2 - \frac{1}{4} \pi d^2$$

$$32. \frac{1}{3} \pi H R^2 + \frac{1}{3} \pi H r^2 + \frac{1}{3} \pi H R r$$

Factor each of the following and then calculate the values:

$$33. 25 \times 41 - 25 \times 39$$

$$34. 3.1416 \times 6 + 3.1416 \times 4$$

$$35. 39.37 \times 46 - 39.37 \times 45$$

$$36. 0.7854 \times 36 - 0.7854 \times 16$$

$$37. 3.1416 \times 80 + 3.1416 \times 48 + 3.1416 \times 32$$

38. One of two boys sold 57 football tickets at 35 cents each, the other sold 23 such tickets. Compute in the quickest way the total amount received.

Collect in parentheses the coefficients of like powers of x :

$$39. ax + mx^2 + bx + nx^2$$

Solution:

$$\begin{aligned} ax + mx^2 + bx + nx^2 &= mx^2 + nx^2 + ax + bx \\ &= (mx^2 + nx^2) + (ax + bx) \\ &= x^2(m + n) + x(a + b) \text{ or} \\ &= (m + n)x^2 + (a + b)x \end{aligned}$$

$$40. mx + cx^2 - nx - dx^2$$

$$41. ax + a^2x^2 - b^2x^2 - bx$$

$$42. ax + bx^3 + cx^2 - dx - ex^2 - fx^3$$

$$43. x^3 + 3x - mx^3 + 3x^2 - 3mx - 3mx^2$$

$$44. a^2x^2 + 2acx + b^2x^2 - 2bcx + c^2 - 2abx^2$$

$$45. px^2 + pqx - x + p - qx^2 - p^2x$$

Add, collecting in parentheses the coefficients of x and y :

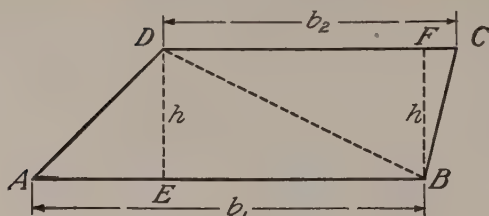
$$46. ax + by$$

$$\underline{dx + cy}$$

$$47. (m + n)x + py$$

$$\underline{mx - py}$$

THE TRAPEZOID



A trapezoid has two parallel sides, called the bases, and two other sides which are not parallel. In the figure, the bases are AB and CD ; the distance DE or FB between the bases is called the altitude of the trapezoid.

Since we know how to find the area of a triangle, the diagonal BD is drawn to divide the trapezoid into two triangles. The base of the triangle ABD is b_1 , the lower base of the trapezoid, and the base of the other triangle BDC is b_2 , the upper base of the trapezoid.

The area of $\triangle ABD = \frac{1}{2} h b_1$

The area of $\triangle BDC = \frac{1}{2} h b_2$

Adding, the area of the trapezoid = ?

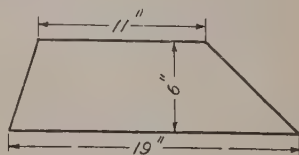
Factor the right member of the resulting formula.

EXERCISE 70

1. Complete the rule: The area of a trapezoid is equal to one-half the sum of the — times the —.

2. Find the area of the trapezoid in the diagram opposite.

3. Find the area of a trapezoid whose upper base is 17 feet 6 inches, lower base 22 feet 6 inches, and whose altitude is 12 feet.



Using the trapezoid formula, fill in the blank spaces (in

Examples 6–9, first substitute the given values in the formula and then solve the resulting equation):

	b_1	b_2	h	A
4.	19"	17"	8"	—
5.	2.5 m.	1.15 m.	1.2 m.	—
6.	15 cm.	13 cm.	—	98 sq. cm.
7.	228'	172'	—	25,000 sq. ft.
8.	—	6"	8"	60 sq. in.
9.	30'	—	10'	260 sq. ft.

10. On squared paper, or with compasses, construct a trapezoid $ABCD$ with right angles at B and C , $BC = 2''$, $CD = 2\frac{1}{2}''$, and $BA = 4''$. Calculate the area of the trapezoid.

11. In the diagram for Example 10, draw a perpendicular to CD through D and thus divide the trapezoid into parts, a rectangle and a triangle. Find the areas of each of these parts, add them, and check with the answer to Example 10.

12. Draw a trapezoid having the same altitude and bases as in Example 10, but (a) with obtuse angles at C and D ;

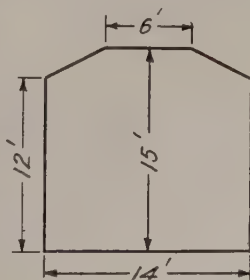
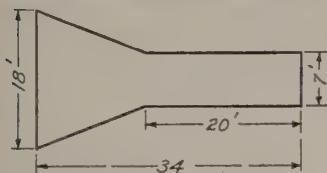
(b) with obtuse angles at D and B . What is the area of each trapezoid?

13. This plan represents a driveway to a double garage. Find the area of the

driveway in square feet. What would it cost at \$2.75 a square yard?

14. Find the floor area of this room which has a bay window.

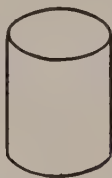
15. Write a formula for the area of the floor of such a room, replacing 14, 15, 12, and 6 by w , l , c , and d respectively.



SURFACES OF RIGHT PRISM AND RIGHT CYLINDER

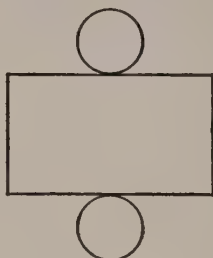
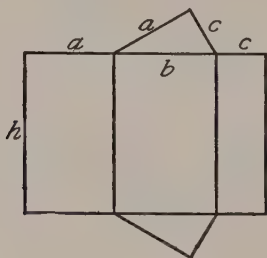
A solid whose ends are equal and parallel polygons, and whose lateral faces are rectangles, is called a right prism. The ends are the bases.

A right cylinder has equal circles for its bases.



If a piece of paper is wrapped around a right prism or a right cylinder, just large enough to cover the lateral surface, and is then cut on a line perpendicular to the bases, unrolled, and laid flat, the paper will be a rectangle.

The pattern, called the development, of the surface of the prism or cylinder consists of a rectangle plus the two bases.



The width of the rectangle equals the — of the prism or cylinder, and its length equals the perimeter of the base.

EXERCISE 71

1. Write a formula for the lateral surface (L) of a right prism or right cylinder in terms of the altitude (h) and the perimeter of the base (P).
2. Write a formula for the lateral surface of a right cylinder: (a) in terms of its altitude and radius (R); (b) in terms of its altitude and diameter (D).

3. Construct on heavy paper or cardboard a pattern for a right prism with a triangular base, the altitude to be 3" and each side of the base 2". Use compasses to construct the triangle. Cut out the pattern, and with small strips of thin paper glue it together to form a right prism.

4. Find the lateral surface of the prism in Example 3.

5. Construct a cylinder from a cardboard pattern.

6. The areas of the three lateral faces of the right prism in the diagram above are ah , —, and —. Fill in the blanks and factor the sum. Write the result in a simpler way by using P . (See Example 1.)

7. A right prism, altitude h , has for its base a polygon whose sides are a, b, c, d , and e . Write the areas of the five rectangles which form the lateral surface; factor the sum.

8. Find a formula for the total surface (S) of a right cylinder by adding the areas of the two bases to the lateral surface. (Use Example 2 (a).)

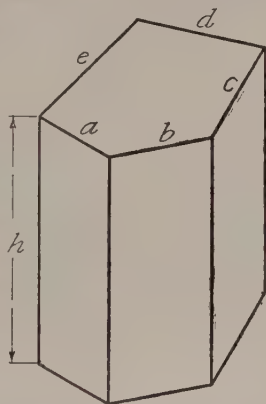
9. Factor the right member of this formula, getting $S = 2\pi$ — ($? + ?$).

10. Find the total surface of a right cylinder in which $R = 7''$ and $h = 26''$.

11. Find the total surface of a right cylinder if the radius of the base is 28 feet and the height is 40 feet.

12. A cylindrical tank is 14' in diameter and 25' high. How much will it cost to paint its entire surface at 40 cents a square yard?

13. Find the number of square inches of tin used in making a cylindrical tin can, whose diameter is 6" and height 4" without considering waste and overlapping.



THE DIFFERENCE OF TWO SQUARES

Typical Example: $a^2 - b^2 = (a + b)(a - b)$

According to a statement in Chapter VI, the difference of the squares of two numbers is equal to the sum of the numbers multiplied by the difference of the numbers.

Example: Factor $c^2 - 9x^2$ and check by multiplication.

$$\begin{aligned} \text{Solution:} \\ c^2 - 9x^2 &= c^2 - (3x)^2 \\ &= (c + 3x)(c - 3x) \end{aligned}$$

$$\begin{array}{r} \text{Check:} \\ c + 3x \\ \underline{c - 3x} \\ c^2 + 3cx \\ - 3cx - 9x^2 \\ \hline c^2 \qquad - 9x^2 \end{array}$$

In order that we may use this method of factoring we must have an expression involving two quantities, each of which is a square, connected by a minus sign. One factor will be the square root of the first quantity plus the square root of the second quantity; the other factor will be the square root of the first quantity minus the square root of the second quantity.

EXERCISE 72

Factor each example if possible. You will find four that cannot be factored.

1. $x^2 - a^2$

2. $c^2 - b^2$

3. $c^2 - a^2$

4. $x^2 - 9$

5. $25 - a^2$

6. $h^2 - 49$

7. $4x^2 - 1$

8. $9a^2 - 1$

9. $4a^2 - 9b^2$

10. $x^2 - 8$

11. $16y^2 - 1$

12. $a^2 - 16$

13. $c^2 - \frac{1}{4}$

14. $b^2 - \frac{1}{25}$

15. $\frac{1}{4}D^2 - 1$

16. $4a^2 + 1$

17. $49a^2 - 4$

18. $25n^2 - 1$

19. $s^2 - \frac{1}{16}$

20. $\frac{1}{4}x^2 - \frac{1}{9}$

21. $25x^2 - 9$

22. $b^2 - 49$

23. $a^2x^2 - 1$

24. $1 - 9x^2$

25. $\frac{1}{4}x^2 + 1$

26. $4x^2 - 25y$

27. $4a^2 - 25h^2$

28. $R^2 - r^2$

Factor each of the following and check:

29. $121x^2 - 49$

30. $64h^2 - 1$

31. $81 - 25b^2$

32. $16y^2 - \frac{1}{9}$

33. $z^2 - 144$

34. $a^6 - 4b^6$

35. $b^4 - 49$

36. $c^4 - 25d^4$

37. $4 - 81x^4$

38. $9a^2b^2 - 25c^2$

39. $144m^4 - 121n^4$

40. $225x^2 - 49$

41. $1 - 169p^2$

42. $196y^4 - 289z^2$

43. $(3m + 5n)^2 - t^2$

Solution:

$$\begin{aligned}(3m + 5n)^2 - t^2 &= [(3m + 5n) + t][(3m + 5n) - t] \\ &= (3m + 5n + t)(3m + 5n - t)\end{aligned}$$

Check:

$$\begin{array}{l|l} (3m + 5n)^2 - t^2 = & 3m + 5n + t \\ (9m^2 + 30mn + 25n^2) - t^2 = & 3m + 5n - t \\ 9m^2 + 30mn + 25n^2 - t^2 & \begin{array}{l} 9m^2 + 15mn + 3mt \\ + 15mn \quad + 25n^2 + 5nt \\ - 3mt \quad - 5nt - t^2 \\ \hline 9m^2 + 30mn \quad + 25n^2 - t^2 \end{array} \end{array}$$

44. $(a + b)^2 - c^2$

45. $(x - y)^2 - g^2$

46. $(m + n)^2 - 16$

47. $(2x + 3y)^2 - 1$

48. $(c + b)^2 - 4x^2$

49. $(s - 2t)^2 - r^2$

50. $(4p + q)^2 - s^2$

51. $(a + b)^2 - \frac{1}{4}$

52. $(x - y)^2 - \frac{1}{9}$

53. $a^2 + 2ab + b^2 - x^2$

54. $a^2 - 2ab + b^2 - c^2$

Separate each of the following into three factors if possible. First take out the common factors if there are any. A few cannot be factored.

55. $3x^2 - 3$

Solution:

$$\begin{aligned} 3x^2 - 3 &= 3(x^2 - 1) \\ &= 3(x + 1)(x - 1) \end{aligned}$$

Check:

$$\begin{array}{r} x + 1 \quad x^2 - 1 \\ x - 1 \quad 3 \\ \hline x^2 + x \quad 3x^2 - 3 \\ -x - 1 \\ \hline x^2 \quad -1 \end{array}$$

56. $ax^2 - a$

57. $5a^2 - 20$

58. $a^2b^2 - 48$

59. $8 - 18y^2$

60. $abx^2 - 9ab$

61. $2x^2 - 16$

62. $4a^2 - 15$

Solution:

$$\begin{aligned} x^4 - y^4 &= (x^2)^2 - (y^2)^2 \\ &= (x^2 + y^2)(x^2 - y^2) \\ &= (x^2 + y^2)(x + y)(x - y) \end{aligned}$$

69. $a^4 - b^4$

70. $q^4 - 1$

71. $b^4 - 16$

72. $3ax^2 - 75a$

63. $2ax^2 - 50a$

64. $\frac{x^2}{4} - \frac{y^2}{9}$

65. $25t - 100t^3$

66. $4w^2 - 100$

67. $3x^2 - \frac{1}{12}$

68. $x^4 - y^4$

Check:

$$\begin{array}{r} x + y \quad x^2 - y^2 \\ x - y \quad x^2 + y^2 \\ \hline x^2 + xy \quad x^4 - x^2y^2 \\ -xy - y^2 \quad +x^2y^2 - y^4 \\ \hline x^2 \quad -y^2 \quad x^4 \quad -y^4 \end{array}$$

73. $5a^4 - 5x^4$ (Four factors)

74. $\pi a^2 - \pi b^2$

75. $81m^4 - 64n^8$

76. $x^8 - b^8$ (Four factors)

Simplify each of the following by factoring and then calculate the value:

77. $76^2 - 24^2$

78. $5.5^2 - 4.5^2$

79. $(5\frac{5}{8})^2 - (3\frac{3}{8})^2$

80. $c^2 - a^2$, if $c = 24$, $a = 1$

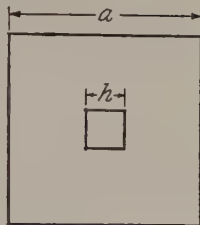
81. $c^2 - a^2$, if $c = 32$, $a = 18$

82. $\frac{1}{4}\pi D^2 - \frac{1}{4}\pi d^2$, if $\pi = 3.1416$, $D = 9$, $d = 1$.

83. A square metal plate has a hole punched out of it. Find in the easiest way the area remaining if $a = 5\frac{3}{4}"$, $h = 1\frac{1}{4}"$.

84. Find the area if $a = 0.47"$, $h = 0.22"$.

85. Open square tin cans are sometimes made from a single square piece of tin by cutting out four equal squares at the corners, as indicated in the diagram. State a formula for the area of the

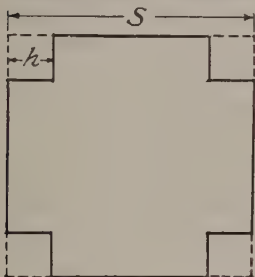


piece of tin remaining, and factor it.

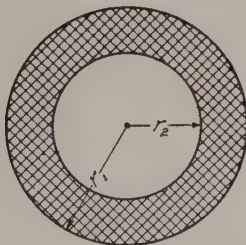
86. Find the area of this tin if $S = 8"$ and $h = 1\frac{1}{2}"$.

87. How many cubic inches will the can made from the tin in Example 86 contain?

88. Write a formula for the width of the bottom of such a can in terms of S and h .



THE ANNULUS



The figure formed by two concentric circles is known as a flat circular ring or an annulus. Explain how we get the formula for its area, $A = \pi r_1^2 - \pi r_2^2$.

EXERCISE 73

1. Separate the right member of this formula into three factors.

2. Use this factored form in finding the area of an annulus if $r_1 = 27''$ and $r_2 = 20''$. Use $\pi = 3\frac{1}{7}$.

3. Find the radii and the area of a flat circular ring if the diameters of the outer and inner circles are $22''$ and $8''$ respectively.

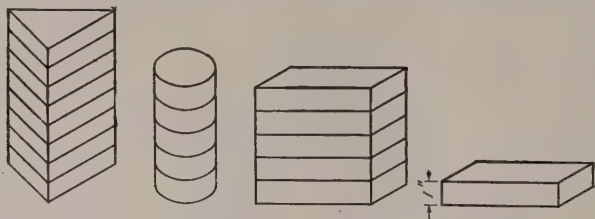
4. Find the area of a circular path $7'$ wide, surrounding a circular pool of water which is $100'$ across.

5. Find the area of an annulus if the diameters of the circles are $3'8''$ and $2'6''$ respectively.

6. The external and internal diameters of a hollow wooden column are $15''$ and $8''$ respectively. What is the area of the cross section?

VOLUME OF PRISM OR CYLINDER

The volume of any solid, such as a prism or a cylinder, which has the same cross section anywhere from the base to the top, is found by multiplying its height by the area of its base.



If by taking cross sections $1''$ apart the solid could be divided into pieces each $1''$ high, there would be as many cubic inches in each piece as there are square inches in the base. Multiplying by the number of pieces, *i.e.* the number of inches in the height, would give the total number of cubic inches in the solid.

EXERCISE 74

1. Write a formula for the volume (V) of a prism or cylinder in terms of its base (B) and altitude (h).

2. Write the same formula for a cylinder in terms of its radius (R) and altitude (h).

3. Show that a cylindrical paint can whose diameter is 7" and height 6" contains exactly one gallon.

4. From a cylinder of radius r_1 , height h , a cylindrical hole is bored out, radius r_2 . Write a formula for the volume (V) remaining and factor the right member of the formula. $V = \pi h$ (?) (?)

5. Find the volume of a hollow cylinder if $r_1 = 22''$, $r_2 = 20''$, $h = 50''$.

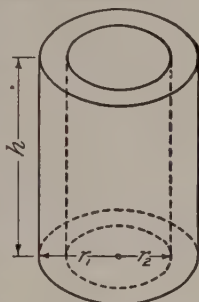
6. The external diameter of a hollow steel shaft is 16" and its internal diameter 12". What is the volume if the shaft is 10' in length?

7. What is the weight of a 12' length of cast-iron pipe 36" in external diameter, 1" thick, the weight of cast iron being $\frac{1}{4}$ lb. per cubic inch?

8. How many cubic inches of copper are there in a tube having an outside diameter $\frac{3}{4}''$, the thickness of the metal being 0.05", and the length of the tube being 60"?

9. The cross section of the rim of a fly wheel is a rectangle $2'' \times 3''$. If the outside diameter of the fly wheel is 26" and the inside diameter 20", what is the volume of the rim?

10. What is the volume of the rim of the fly wheel in Example 9 if the inside diameter is 22", outside diameter 26"?



TRINOMIALS THAT ARE PRODUCTS OF TWO BINOMIALS

In Chapter VI we learned how to find mentally the products of two binomials such as $(5x + 3)(2x + 1)$ and $(3a - 2b)(4a + 3b)$. What are the two products?

If we wish to reverse this process and to discover the factors of an expression such as $3x^2 + 17x + 10$, we know that

- (1) the product of the first terms of the binomials must be $3x^2$,
- (2) the product of the second terms of the binomials must be 10, and
- (5) the sum of the cross-products must give the middle term $17x$.

The factors of $3x^2$ are $3x$ and x ; these are the first terms of the binomials we are trying to find.

The factors of 10 are either 1 and 10, or 2 and 5; one of these pairs of factors gives the second terms of the binomials.

The possible arrangements are as follows:

$$\begin{array}{r} 3x+1 \\ \times \\ x+10 \\ \hline 31x \end{array}$$

$$\begin{array}{r} 3x+10 \\ \times \\ x+1 \\ \hline 13x \end{array}$$

$$\begin{array}{r} 3x+5 \\ \times \\ x+2 \\ \hline 11x \end{array}$$

$$\begin{array}{r} 3x+2 \\ \times \\ x+5 \\ \hline 17x \end{array}$$

In order to select the correct set of factors it is necessary to find the sum of the cross-products in each case, as above. Since the arrangement at the right gives $17x$ as the sum of the cross-products it is clear that

$$3x^2 + 17x + 10 = (3x + 2)(x + 5) \quad \text{Ans.}$$

Check:

$$\begin{array}{r} 3x + 2 \\ x + 5 \\ \hline 3x^2 + 2x \\ + 15x + 10 \\ \hline 3x^2 + 17x + 10 \end{array}$$

Example: Factor $2x^2 + 15x + 7$.

We have $2x^2 = 2x \cdot x$, therefore $2x$ and x are the first terms of the binomials.

Also $7 = 1 \times 7$, therefore 1 and 7 are the second terms of the binomials.

Write down the possible arrangements and find the sum of the cross-products:

$$\begin{array}{r} 2x+7 \\ \times \\ x+1 \\ \hline 9x \end{array} \qquad \begin{array}{r} 2x+1 \\ \times \\ x+7 \\ \hline 15x \end{array}$$

$$\therefore 2x^2 + 15x + 7 = (2x + 1)(x + 7) \quad \text{Ans.}$$

Example: Factor $6x^2 - 13xy + 5y^2$.

The plus sign for the third term shows that the factors of $5y^2$ must be of like sign, and the minus sign for the second term shows that this sign must be minus.

Since $6x^2 = 6x \cdot x$ or $3x \cdot 2x$, therefore the first terms of the binomials are either $6x$ and x , or $3x$ and $2x$.

Also $5y^2 = (-5y)(-y)$; therefore the second terms of the binomials are $-5y$ and $-y$.

The possible arrangements and the sum of the respective cross-products are:

$$\begin{array}{r} 6x-5y \\ \times \\ x-y \\ \hline -11xy \end{array} \qquad \begin{array}{r} 6x-y \\ \times \\ x-5y \\ \hline -31xy \end{array} \qquad \begin{array}{r} 3x-y \\ \times \\ 2x-5y \\ \hline -17xy \end{array} \qquad \begin{array}{r} 3x-5y \\ \times \\ 2x-y \\ \hline -13xy \end{array}$$

$$\therefore 6x^2 - 13xy + 5y^2 = (3x - 5y)(2x - y) \quad \text{Ans.}$$

In doing these examples it is a good plan not to use "scratch paper." Set down neatly each arrangement as you try it, writing the sum of the cross-products which you obtain mentally. This affords a complete record of your work, and may save you from trying the same arrangement twice. It will also show your teacher any false steps you may have taken or obstacles you have encountered. Do not expect the very first trial to be successful.

EXERCISE 75

1. $5a^2 + 11a + 2$
2. $2b^2 + 5b + 3$
3. $3E^2 + 4E + 1$
4. $7x^2 + 8x + 1$
5. $3D^2 - 5D + 2$
6. $11t^2 - 23t + 2$
7. $3m^2 - 16m + 5$
8. $7x^2 - 8x + 1$
9. $10x^2 + 17x + 7$
10. $12p^2 - 7p + 1$
11. $6a^2 - 11a + 3$
12. $7m^2n^2 - 16mn + 4$
13. $20q^2 + 12q + 1$
14. $9z^2 + 19z + 10$
15. $9z^2 - 91z + 10$
16. $21r^2 - 38r + 16$
17. $3s^2 + 2s - 5$
18. $10x^2 + x - 3$
19. $6h^2 - h - 35$
20. $12c^2 - 8c - 15$
21. $12m^2 + 8m - 15$
22. $2x^2 + 7xy + 3y^2$
23. $5x^2 - 8xz + 3z^2$
24. $6b^2 + 17bx - 14x^2$
25. $9a^2 + 9ay - 10y^2$
26. $6b^2 + 11bc - 10c^2$
27. $15a^2b^2 - 11ab - 14$
28. $2R^2 + Rh - 3h^2$
29. $6g^2 - 19gf - 77f^2$
30. $a^4 + 3a^2 + 2$
31. $4 - 8a - 5a^2$
32. $6 + x - 2x^2$
33. $7y^4 - 50y^2 + 7$
34. $7t^4 + 48t^2 - 7$
35. $6 + 5m^2 - 6m^4$
36. $5 + 3a - 2a^2$
37. $10 - p - 2p^2$
38. $4a^2 + 4ac - 99c^2$
39. $6 + 11a - 10a^2$
40. $8 - 6b^2 - 9b^4$
41. $36x^2 - 97xy + 36y^2$
42. $24 + 55h - 24h^2$
43. $4a^2 + 12a + 9$
44. $9h^2y^2 + 30hyz + 25z^2$
45. $a^2 + 7a + 10$
46. $x^2 + 8x + 15$
47. $y^2 + 5y + 6$
48. $m^2 + 9m + 14$
49. $r^2 + 9r + 18$
50. $x^2 - 9x - 22$

Trinomials like the preceding six, in which the coefficient of the first term is 1, are more readily factored than are the others. Thus in Example 45, to factor $a^2 + 7a + 10$, there is no choice as to the first terms of the factors, since $a^2 = a \cdot a$, and we have only to select that pair of factors of 10 whose sum is 7, — namely, 5 and 2. Therefore the factors of $a^2 + 7a + 10$ are $a + 5$ and $a + 2$.

51. What pair of factors of 30 has the sum 13? 11? 17?
52. Factor $m^2 + 11m + 30$.
53. Factor $s^2 - 13s + 30$.

In like manner find the factors of:

- | | |
|-------------------------|--------------------------|
| 54. $y^2 - 15y + 56$ | 61. $x^2 + 2x - 48$ |
| 55. $h^2 - 14h + 33$ | 62. $m^4 - 9m^2 + 14$ |
| 56. $x^2 + 6x - 55$ | 63. $h^2 - hx - 72x^2$ |
| 57. $y^4 - 10y^2 + 21$ | 64. $t^2 + 48tw - 49w^2$ |
| 58. $b^2 - 6b + 8$ | 65. $t^2 - 48ta - 49a^2$ |
| 59. $a^2n^2 + 3an - 10$ | 66. $a^4 + a^2 - 42$ |
| 60. $c^4 - 2c^2 - 15$ | 67. $x^6 + 12x^3 + 27$ |

Factor and check:

- | | |
|---|-------------------------------|
| 68. $4a^2 + 12a + 9$ | 74. $9p^2 - 30p + 25$ |
| 69. $16x^2 - 8x + 1$ | 75. $49 - 28a + 4a^2$ |
| 70. $4b^2 + 28b + 49$
(See Note below) | 76. $9x^2 - 12xy + 4y^2$ |
| 71. $36c^2 - 60cn + 25n^2$ | 77. $9x^4 - 30x^2z^2 + 25z^4$ |
| 72. $m^2 - 12m + 36$ | 78. $100 - 140b + 49b^2$ |
| 73. $4h^2 - 20h + 25$ | 79. $9a^2 - 60a + 100$ |

NOTE.—When the two factors are the same, the trinomial is revealed as the square of a binomial. No special directions are needed for factoring such trinomials, but the work done in Chapter VI helps one to recognize them at sight. For further study see Chapter XII.

Separate each of the following into three factors and check:

- | | |
|------------------------------|--------------------------------|
| 80. $2x^2 + 14x + 20$ | 87. $3vt - 9v + 6vt^2$ |
| 81. $54 + 3x^2 - 27x$ | 88. $98 + 35m - 7m^2$ |
| 82. $ax^2 + 3ax - 10a$ | 89. $6a^2bc + abcx - bcx^2$ |
| 83. $a^2b^2 - 2ab^2 - 15b^2$ | 90. $-8 + 24a - 18a^2$ |
| 84. $2a^2c - 12ac - 54c$ | 91. $42b^2 + 11cb^2 - 3c^2b^2$ |
| 85. $6x^2 - 9 - 3x$ | 92. $-9b + 12ab - 4a^2b$ |
| 86. $10m^2 - 25m - 15$ | 93. $2\pi r^2 + 5\pi r + 2\pi$ |

A REVIEW

The following directions give you a plan of attack for factoring any expression:

First: Try to find a common factor.

Second: Count the terms; if there are two terms, is it the difference of two squares?

Third: If there are three terms, try the cross-products method.

Do not leave any example until you have examined carefully each of your binomial or trinomial factors and are sure that they are prime.

EXERCISE 76

Find the factors of:

- | | |
|-----------------------------------|---|
| 1. $m^2 - x^2$ | 21. $36 y^2 - 120 y + 100$ |
| 2. $x^2 - 19 x + 88$ | 22. $22 a^3 + 33 a^2 - 99 a$ |
| 3. $9 a^2 - 42 at + 49 t^2$ | 23. $\frac{1}{2} \pi D^2 - \frac{1}{2} \pi d^2$ |
| 4. $a^3 - ab^2$ | 24. $\frac{1}{3} \pi R^2 H - \frac{1}{3} \pi r^2 H$ |
| 5. $6 x^3 - 9 x^2 - 6 x$ | 25. $3 a^2 - 27$ |
| 6. $2 x^2 - 2 x - 112$ | 26. $5 t^2 + 45$ |
| 7. $3 c^2 - 48$ | 27. $m^6 - n^2$ |
| 8. $7 x^2 + 35 xy + 35 y^2$ | 28. $2 x^2 + 4 xz + 8 z^2$ |
| 9. $81 c^2 - 121 b^2$ | 29. $vt + 16 t^2$ |
| 10. $9 x^2 - 24 xc + 16 c^2$ | 30. $m^3 s - ms^3$ |
| 11. $a^3 x - a^2 x - 42 ax$ | 31. $7 a^4 - 7 n^2$ |
| 12. $3 t^4 - t^2 s^2 - 2 s^4$ | 32. $-12 x^2 + 12 x - 3$ |
| 13. $14 mn^4 - 28 mn^3 + 42 mn^2$ | 33. $\pi r_1^2 + \pi r_1 r_2 + \pi r_2^2$ |
| 14. $\pi R^2 - 4 \pi r^2$ | 34. $x^2 - x + \frac{1}{4}$ |
| 15. $12 a^2 - 10 ab - 12 b^2$ | 35. $a^4 - 10 a^2 + 9$ |
| 16. $16 x^2 - 112 xy + 196 y^2$ | 36. $3 ab + 3 ac$ |
| 17. $3 pq^2 - 9 pq - 30 p$ | 37. $121 x^2 - 66 xy + 9 y^2$ |
| 18. $121 h^2 - 16$ | 38. $rs^2 + rs - 30 r$ |
| 19. $9 a^2 b^2 c^2 - 1$ | 39. $(a + b)^2 - c^2$ |
| 20. $2 mn^2 - 2 mn - 24 m$ | 40. $p^2 q + pq^2$ |

41. $12x^2 - 26x + 12$

47. $\frac{1}{2}a^2 + \frac{1}{2}ac + \frac{1}{2}c^2$

42. $ma^2 - mb^2$

48. $18t^2 - 12t + 2$

43. $4x^2 - 36$

49. $3a^2bc + 3ab^2c + 3abc^2$

44. $2at^2 - 8at - 42a$

50. $169z^2 - 1$

45. $7c^2 + 14cd + 7d^2$

51. $2m^2x - mnx - 3n^2x$

46. $d^4 + 4d^2 - 5$

52. $(x + c)^2 - 9$

53. $a^2 + 2ab + b^2 - 16$

54. Any expression which is in factored form appears as a product. Products are often more easily understood and calculated than are sums, and knowledge of factoring becomes useful in order to change from a sum to a product. Write the sum of ax and bx as a product.

55. Write the sum $r^2 + r^3 + r^4$ as a product.

56. Write the product $(a + 7)(a + 2)$ as the sum of three terms.

57. Write $c^2 - b^2$ in product form and calculate its value if $c = 97$ and $b = 3$.

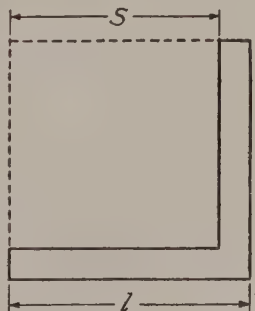
58. Remove parentheses from $x^2 + 3(4x + 9)$ and by factoring write it as a product. Check by substituting $x = 2$.

59. Transform $x(x + 5) - 14$ into a product and check.

60. A figure very famous in olden times was formed by cutting out a square from the corner of a larger square. Write a formula for its area in the form of a product.

61. Find the area of the part of the square remaining if $l = 24''$ and $s = 21''$.

62. Steel bars, known as "angles," have a cross section as indicated in the diagram for Example 60. Find the cross section area if $l = 6''$ and $s = 5\frac{1}{2}''$.



63. Derive a formula for the area of an annulus in terms of the diameters of the circles.

64. A right cylinder is often called a "cylinder of revolution." Find R , h , and V of the cylinder formed by revolving a square whose side is 10" about one of its sides as an axis.

65. A rectangle $14'' \times 8''$ is revolved about one of its longer sides as an axis. Find the volume of the resulting cylinder.

66. What is the volume of the cylinder formed by revolving a rectangle $14'' \times 8''$ about one of its shorter sides as an axis? Compare your answer with the answer to Example 65.

67. Find the lateral surfaces of the cylinders in Examples 65 and 66 and compare them.

68. Compare the lateral surfaces of the two different cylinders formed by revolving a rectangle $l'' \times w''$ about one of its sides as an axis.

69. Derive a formula for the volume of a hollow cylinder or pipe in terms of its altitude and the diameters of the circles.

70. How many cubic yards of concrete are there in a concrete silo, outside diameter 20', inside diameter 18', height 30'?

71. How many bushels would the silo contain, allowing $1\frac{1}{4}$ cubic feet to the bushel?

72. Find a formula for the total area (A) of the four walls of a room $l' \times w'$, h' high. Express in factored form.

73. Use the formula found in Example 72 to obtain the cost of painting the walls of a room $12' \times 15'$, 9' high, at 36 cents a square yard.

74. Draw with compasses and ruler a quadrilateral $ABCD$,— $AB = 4''$, $BC = 3''$, diagonal $AC = 4''$, $CD = 3''$, $DA = 2''$. Let fall a perpendicular from D to AC and one from B to AC . Measure these two perpendiculars; find the areas of the two triangles, and so find the area of the quadrilateral.

75. Find a formula for the area (A) of the quadrilateral if AC is d , and the two perpendiculars are h_1 and h_2 respectively. Express the right member of the formula as a sum; as a product.

76. Use the formula obtained in Example 75 to get the area of a quadrilateral in which one diagonal is $15'$ and the perpendiculars to it from the other vertices are $10'$ and $7'$ respectively.

77. Find the area of a quadrilateral $ABCD$ if $AD = 17'$, $DC = 25'$, $CB = 21'$, and the angles at D and C are right angles.

A TEST ON FACTORING

Find the prime factors in each example. Check by multiplication.

Time, 25 minutes.

- | | |
|-----------------------|----------------------------|
| 1. 165 | 7. $12n^2 + 23n + 10$ |
| 2. $an + nz$ | 8. $b^2 - bf - 2f^2$ |
| 3. $x^2 + 11x + 18$ | 9. $4y^2 + 11y - 3$ |
| 4. $r^2 - 100$ | 10. $6r^2 - rs - 15s^2$ |
| 5. $4x^2 - 4xy + y^2$ | 11. $D^3 - D$ |
| 6. $c^2 + cr$ | 12. $a^2 + 2ab + b^2 - 25$ |
13. Write $10 + w(7 + w)$ as a trinomial. Write it in the form of a product.

(The median class score should probably be 9 or above, counting each correct solution $\frac{2}{3}$, each check $\frac{1}{3}$, and each incorrect or incomplete solution 0.)

A SECOND TEST ON FACTORING

Find the prime factors in each example. Check by multiplication.

Time, 25 minutes.

- | | |
|----------------------|-----------------------------|
| 1. 66 | 7. $6p^2 - pq - 12q^2$ |
| 2. $w^2 + 12w + 20$ | 8. $2R + \pi R$ |
| 3. $f^2 + 2fn + n^2$ | 9. $y^2z^2 - 2yz - 8$ |
| 4. $3c^2 + 14c + 8$ | 10. $N^3 - N^2 - 90N$ |
| 5. $4a^2 - p^2$ | 11. $9l - 4l^3$ |
| 6. $10x^2 - 19x + 6$ | 12. $x^2 + 2ax + a^2 - b^2$ |
13. What is the highest common factor of the three terms of the trinomial $6a^2b + 12ab^2 + 8b^3$? Factor the trinomial.

ADDITIONAL AND REVIEW EXAMPLES

On the First Eight Chapters of the Book

Evaluation of Formulas:

1. When $t = 20$ and $u = 1$, what is the value of $2tu + u^2$?
2. What is the difference between the values of n^2 and $2n$, when $n = 5$?
3. If $a = -2$ find the value of $a^2 - 4a + 4$.
4. Which is greater, $2ab$ or $a^2 + b^2$, if $a = 5$, $b = 3$?
if $a = 1$, $b = 6$?
5. Using the tables on page 377, find the value of $h^3 + w^3$ when $h = 34$ and $w = 43$.
6. From the same tables find the value of $\sqrt{h}$, when $h = 34$.
7. From the formula $i = prt$ find i if $p = \$504$, $r = 6\%$, $t = \frac{36}{360}$.
8. Using the formula $S = vt + \frac{1}{2}at^2$ calculate the value of S when $v = 44$, $t = 4$, and $a = 5$.
9. Given the formula $A = p + prn$, find A if $p = \$80$, $r = 4\%$, and $n = 3$.
10. Using the formula $T = 2\pi R(R + h)$ find the total surface of a cylinder having $R = 50''$ and $h = 10''$. ($\pi = 3.14$.)
11. The surface of a sphere is obtained by the formula $S = 4\pi R^2$. Find the surface of a tennis ball $2''$ in diameter.
12. Find the values of the trinomial $x^2 + 2x + 3$ for each value of x from -3 to $+3$, and tabulate:

x	$x^2 + 2x + 3$	ANSWER
-3	$9 - 6 + 3 =$	6
-2		
-1		
0		
etc.		

Geometric Problems:

13. Common bricks are $8'' \times 4'' \times 2''$. What is the volume of a brick?

14. If only 22 bricks are used for each cubic foot of masonry, what fractional part of the cubic foot is taken up by the bricks? What fractional part is taken up by the mortar?

15. How many faces has a brick? how many edges? how many corners?

16. What is the total surface of a common brick in square inches?

17. On squared paper make a drawing to represent a triangle, base $12''$, altitude $9''$. What is the area of the triangle?

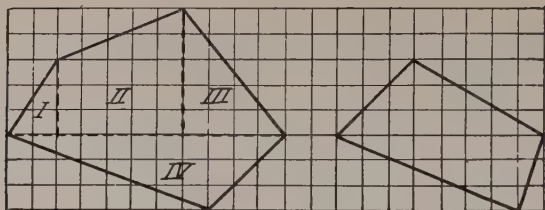
18. Make a drawing to represent a parallelogram having the same base and altitude as the triangle in the preceding example. What is the area of the parallelogram?

19. The cross-section of a railway embankment is in the form of a trapezoid, bases $12'$ and $30'$, altitude $7'$. What is the area of the trapezoid? Make a drawing of the trapezoid on squared paper.

20. Draw any triangle, on plain paper. Let fall the altitude, using compasses. Take measurements, and find the area of the triangle.

21. On plain paper construct very carefully with compasses a rectangle whose base shall be $4\frac{1}{2}''$ and whose area shall be 9 square inches.

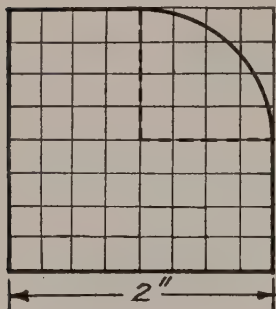
22. Using compasses, construct a square that shall have the same area as the rectangle in the preceding example.



23. Find the area of the 5-sided figure in the diagram by getting successively the areas of I, II, III, and IV.

24. In a similar way find the area of the quadrilateral.

25. A cylindrical paint can is $8\frac{1}{4}$ " high and 6" in diameter, inside dimensions. How much more or less does it hold than a gallon, 231 cubic inches?



26. On squared paper draw a square 2" on a side, with one-fourth of the square replaced by a quarter circle.

27. Find the area of the figure you have just drawn. ($\pi = 3.14$.)

28. Find the perimeter of the figure.

*29. Write formulas for the perimeter and area of such a figure if the side of the square is $2R$.

The Four Fundamental Operations:

30. Write five consecutive numbers of which n is the smallest, and add them. Check when $n = 4$.

31. Write four consecutive even numbers of which $2n + 4$ is the largest, and find their sum. Check when $n = 5$.

32. Write the square of $a + 5$ and the square of $a - 5$, and add them.

33. Find the difference of the squares in the preceding exercise.

34. Add $3x + 2y - 5$, $2x - y + 1$, and $3y + 4x - 6$. Check when $x = 1$, $y = 1$.

35. Add $r^2 - 3rx - 2x^2$, $5x^2 - 6rx + r^2$, $3x^2 - 4r^2$, and $r^2 + rx$. Check.

36. From $10t + u - 3$ subtract $t + u - 5$. Check.

37. Subtract $3x - y - 2z$ from $x + y - 7z$. Check.

38. What is the square of the difference of $a + 2b$ and $a - 2b$?

39. Multiply $a^2 + 5a - 5$ by $2a - 1$. Check when $a = 2$.

40. Find the product of $n^2 + 2n + 4$ and $n - 2$. Check.

41. Divide $3c^2 - 10c + 3$ by $c - 3$. Check by multiplication.

42. Divide $x^3 - 8a^3$ by $x - 2a$. Check by multiplication.

43. Is $10 - (2k + 3) - 7$ equal to $2k$?

44. Remove parentheses and simplify: $x^2 - (x^2 - 2x + 1)$.

Perform the indicated operations and simplify:

45. $3(a + b) - 2(a - b)$

46. $(D + 1)(D - 1) - (D + 1)^2$

47. $5a - [4a - (3a - 2)]$

48. $(n + 3)(n - 2) - (n + 2)(n - 3)$

Equations and Problems:

49. $3(x + 3) = 5(x + 2) - 1$

50. $l + l - 6 + l + l - 6 = 80$

51. $0.1r = 12$

52. $.08x = .07x + 57$

53. $.06p - .05(p - 600) = 48$

54. $(2n + 1)^2 = (2n - 1)^2 + 32$

55. $s^2 - s(s - 2) = 14$

56. $36 - (x + 1)(x + 3) = 21 - x(5 + x)$

57. Find l from the formula $p = 2l + 2w$ if $p = 74$ and $w = 12$.

58. In the same formula find w if $p = 34$ and $l = 11\frac{1}{2}$.

59. From the formula $l = a + (n - 1)d$ find n if $a = 7$, $l = 40$, and $d = 3$.

60. Find a from the formula $v = v_0 + at$ if $v = 170$, $v_0 = 10$, and $t = 5$.

61. I am thinking of a number which is 14 less than three times itself. What is the number?

62. Either arm of an isosceles triangle is four times as long as the base, and the perimeter of the triangle is 180 centimeters. Find the base.

63. Find w from the formula $E = \frac{wv^2}{64}$ if $v = 4$ and $E = 100$.

64. Find R from the formula $F = \frac{wv^2}{gR}$ if $F = 20$, $w = 10$, $v = 40$, and $g = 32$.

65. What is the value of q in the formula $D = dq + r$ when $D = 75$, $d = 9$, and $r = 3$?

66. The perimeter of a rectangle is $30\frac{5}{8}$ ". Find its dimensions if it is $7\frac{1}{2}$ " longer than it is wide.

67. A rectangle of width w inches is 3 times as long as it is wide. What is its perimeter? What is the side of a square having the same perimeter? What would be the value of w in order that the square should contain 16 square inches more than the rectangle?

68. A bottle full of medicine cost \$1.10. If the medicine cost \$1 more than the bottle, what was the cost of the latter?

69. Find two numbers which differ by 3 such that if the smaller is added to 4 times the larger the sum is 32.

70. A house and a lot together cost \$9600. If the cost of the house was \$1200 more than 5 times the cost of the lot, what was the cost of the lot?

71. I am thinking of two numbers whose sum is 23. If twice the larger number be subtracted from 40, the result

is the same as if six times the smaller be subtracted from 50. What are the numbers?

72. The length of a single tennis court exceeds the width by 51 feet. A double court is 9 feet wider but of the same length; its area exceeds that of the single court by 702 square feet. Find the dimensions of the single court.

73. A and B have equal sums of money. If A gives B \$150, then B will have three times as much as A will have. How many dollars has each now?

74. Mr. Gardner said to Mr. Sheehan, "You have twice as much money as I." Mr. Sheehan replied, "No, for I have lost \$5; but if you will pay me the \$15 that you owe me I shall have four times as much as you will." How much has Mr. Gardner?

75. A square room measures 5 feet longer on each side than does a rug. How long is the room if the rug covers all except 105 square feet of the floor?

76. A certain piece of bronze weighing 690 lb. contains twice as much tin as zinc, and $4\frac{1}{4}$ times as much copper as tin. How many pounds of each metal are in the piece?

Making Formulas:

77. If a child enters the first grade at age 6 and is promoted at the end of each year, how old will he be in the second grade? in Grade 3? Grade 5?

78. Make a table giving in one column a list of the grades from 1 to 8, and in the other column the child's age.

79. State a formula for the number of years in his age (y) in terms of the number of his grade (g).

80. In what grade will he be when 10 years old? State a formula for g in terms of y .

81. What is the difference of the squares of 6 and 5?

82. Is this equation correct or not?

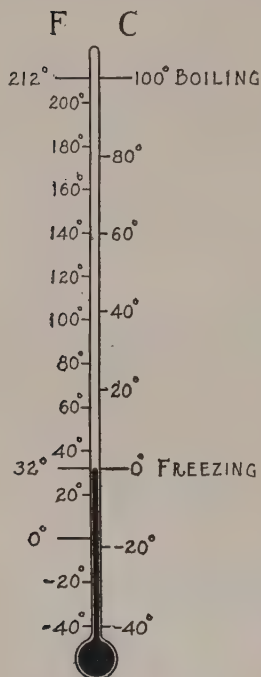
$$11^2 - 10^2 = 11 + 10$$

83. Write two consecutive numbers of which n is the smaller. Show that the difference of the squares of two consecutive numbers is always equal to the sum of the numbers. Verify this fact when $n = 50$.

84. In the table of squares of numbers from 1 to 50, find the difference between each two successive values of n^2 from $n = 1$ to $n = 5$, and from $n = 46$ to $n = 50$.

85. Write three consecutive numbers of which m is the middle one. Square each of the numbers and find the sum of the squares. Verify by substituting $m = 7$.

86. After doing Example 85, copy and complete this statement: "The — of the — of any three consecutive numbers exceeds — times the square of the middle number by —."



The Fahrenheit thermometer scale in common use is marked 32° at the freezing temperature and 212° at the boiling temperature; the Centigrade thermometer is marked 0° and 100° , respectively, for the same temperatures. Thus 100 degrees on the C. scale correspond to 180 degrees on the F. scale, or

$$\begin{aligned} 100 \text{ C. degrees} &= 180 \text{ F. degrees} \\ 5 \text{ C. degrees} &= 9 \text{ F. degrees} \\ 1 \text{ C. degree} &= \frac{9}{5} \text{ F. degrees} \end{aligned}$$

87. The diagram shows a thermometer (with the temperature at the freezing point) marked on one side with the F. scale and on the other with the C. scale.

(a) From the diagram tell what the C. reading would be if the temperature was 140° Fahrenheit.

(b) Is there any temperature at which the F. and C. readings are the same?

88. The rule for changing a Centigrade reading to a Fahrenheit is stated: "To find F multiply C by $\frac{9}{5}$ and add 32." Express this rule as a formula.

89. Fill in the blank spaces, finding the value of F for each given value of C :

C	0°	20°	40°	60°	80°	100°	-20°
F	32°	68°					-4°

Check each value obtained by locating roughly in the diagram the points on the F. scale opposite the given values of C.

90. The subject of the formula can be changed, giving $C = \frac{5}{9}(F - 32)$. State this as a rule.

91. By this formula or rule find C

(a) when F is 59.

(b) when F is -13 .

(c) when F is -40 .

Check your three answers by the diagram.

92. If a teacher's salary starts at \$1200 and increases \$100 at the end of each year, what will the salary be after one year of service? after two years? after five years?

93. Write a formula for the teacher's salary (s) after n years of service.

94. Draw a graph showing the teacher's salary after each year of service up to 10 years.

95. Draw a broken line graph showing the amount of U. S. income tax a married man must pay, from the accompanying tabulation:

Income	\$2500	\$3000	\$4000	\$5000	\$6000	\$8000	\$10000	\$12000
Tax	0	20	60	100	160	340	520	720

96. A formula for the tax on incomes up to \$5000 is $I = .04(x - 2500)$. Use this formula to calculate the tax on an income of \$3500. Check your result with the graph.

97. State as a rule the formula in the preceding example.

98. A battleship starts out with 5000 tons of coal. It burns 300 tons a day. How much coal will be left after 5 days? after 16 days?

99. Write a formula for the amount of coal remaining after d days.

100. Try to see how each of these series of numbers is formed, and write the next three terms:

(a) 2, 4, 6, 8, —, —, —

(b) 13, 9, 5, 1, —, —, —

(c) 5, 10, 20, 40, —, —, —

(d) 17.3, 1.73, .173, —, —, —

(e) $\frac{3}{8}$, 1, $1\frac{5}{8}$, —, —, —

(f) 1, 4, 9, 16, —, —, —

(g) 3, 0, -3, —, —, —

*101. Write a formula for the n th term of series a ; of series f ; and of any other series that you can.

CHAPTER IX

FRACTIONS

A fraction is simply an indicated division. The dividend is called the numerator and the divisor the denominator. Thus $\frac{8}{5}$ means 8 divided by 5; 8 is the numerator and 5 the denominator; $\frac{a}{b}$ means a divided by b and is commonly read “ a over b .”

The numerator of a fraction may be any number whatsoever; the denominator may be any number except 0. The numerator and the denominator are called the terms of the fraction.

The fundamental principle of fractions is:

Multiplying or dividing both numerator and denominator of any fraction by the same number does not change the value of the fraction.

Fractions which have the same value are called equivalent fractions.

Oral Exercise

1. Show why it is that $\frac{20}{24} = \frac{5}{6}$.
2. By what principle may we change $\frac{6}{8}$ to $\frac{3}{4}$?
3. What principle do we use in changing $\frac{a}{b}$ to $\frac{am}{bm}$?

In each of the following examples tell what was done to the numerator and to the denominator of the first fraction to produce the second fraction:

4. $\frac{3}{8}, \frac{6}{16}$

5. $\frac{14}{18}, \frac{7}{9}$

6. $\frac{bx}{ax}, \frac{b}{a}$

10. $\frac{a}{b}, \frac{a^2b}{ab^2}$

7. $\frac{m}{n}, \frac{mp}{np}$

11. $\frac{x}{a+b}, \frac{2x}{2a+2b}$

8. $\frac{ab}{xy}, \frac{abc}{cxy}$

12. $\frac{6p}{3p-3q}, \frac{2p}{p-q}$

9. $\frac{m^2}{mn}, \frac{m}{n}$

13. $\frac{4x+4y}{8x-8y}, \frac{x+y}{2x-2y}$

14. Is $\frac{12}{32}$ equal to $\frac{3}{8}$?

15. Is $\frac{a^2}{a}$ equal to a ?

16. Does $\frac{x}{y}$ equal $\frac{x^2}{y^2}$? Why?

17. Is this equation true: $\frac{a}{b} = \frac{3a}{3b}$? Test your answer when $a = 2$ and $b = 3$.

18. Does $\frac{a}{b}$ have the same value as $\frac{a+3}{b+3}$? Test when $a = 1$ and $b = 2$.

19. Change $\frac{2}{3}$ to a fraction whose denominator is 9.

20. Change $\frac{1}{4}$ to sixteenths.

21. Change $\frac{1}{b}$ to a fraction having bc as a denominator.

22. Write $\frac{x}{y}$ as an equivalent fraction with denominator y^2 .

Change each of the following fractions to a fraction which has the denominator indicated:

23. $\frac{2p}{3q}$, denominator $6q$

24. $\frac{8}{16}$, denominator 2

25. $\frac{12}{15}$, denominator 5.
26. $\frac{3ab}{6ac}$, denominator $2c$
27. $\frac{4x}{2(x-y)}$, denominator $x-y$
28. $\frac{1}{2}, \frac{1}{5}$, common denominator 10
29. $\frac{1}{p}, \frac{1}{q}$, common denominator pq
30. $\frac{5}{6}, \frac{7}{10}$, common denominator 60
31. $\frac{a}{x}, \frac{b}{y}$, common denominator xy
32. $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$, common denominator 30
33. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, common denominator abc
34. $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}$, common denominator 16
35. $\frac{a}{b}, \frac{b}{c}, \frac{a}{c}$, common denominator bc
36. Change $\frac{75}{100}$ to an equivalent fraction having the numerator 3.
37. Change $\frac{bc}{x}$ to a fraction having the numerator abc .

In the following examples, the fractions are equal to each other. Fill in the term that is missing:

38. $\frac{7}{8} = \frac{?}{16}$

40. $\frac{?}{9} = \frac{21}{27}$

39. $\frac{4}{16} = \frac{8}{?}$

41. $\frac{15}{24} = \frac{5}{?}$

42. $\frac{a}{b} = \frac{ax}{?}$

45. $\frac{3 abx}{6 acy} = \frac{bx}{?}$

43. $\frac{pq}{qr} = \frac{?}{r}$

46. $\frac{x}{mx + nx} = \frac{1}{?}$

44. $\frac{?}{2 m} = \frac{3 mn}{6 m}$

47. $\frac{2(a + b)}{3(a + b)} = \frac{?}{3}$

REDUCTION OF FRACTIONS TO LOWEST TERMS

A fraction is said to be reduced to lowest terms when the numerator and the denominator have no common factor.

Oral Exercise

Reduce to lowest terms:

1. $\frac{16}{24}$

9. $\frac{9 x^2}{12 xy}$

2. $\frac{18}{42}$

10. $\frac{a^2b}{ab^2}$

3. $\frac{3 a}{6 b}$

11. $\frac{\pi R h}{\pi R^2}$

4. $\frac{2 m}{3 m}$

12. $\frac{3 a^3}{15 a^2}$

5. $\frac{6 ax}{4 ay}$

13. $\frac{18 x^3y^2}{6 x^2y^3}$

6. $\frac{\pi D}{\pi d}$

14. $\frac{2(a - b)}{3 c(a - b)}$

7. $\frac{2 \pi r}{2 \pi R}$

15. $\frac{(x - 2) (a + b)}{(x - 2) (a - b)}$

8. $\frac{14 w^2}{7 w}$

16. $\frac{a + b}{5 a + 5 b}$



17. Name, in order, the divisions on a ruler which shows eighths, from $\frac{1}{8}$ " to 1". Express each fraction in its lowest terms.

18. Name, in order, the divisions on a ruler which shows sixteenths, from $\frac{1}{16}$ " to $\frac{1}{2}$ ". Express each fraction in its lowest terms.



It is usually necessary to factor numerator or denominator, or both, in order to reveal common factors which may be cancelled. Always make sure the numerator and the denominator are in factored form before attempting to reduce fractions to lowest terms.

Example: Reduce $\frac{x^2 - 4}{x^2 - 5x + 6}$ to lowest terms and check by letting $x = 4$.

Solution:

$$\frac{x^2 - 4}{x^2 - 5x + 6} = \frac{(x + 2) \cancel{(x - 2)}}{(x - 3) \cancel{(x - 2)}} = \frac{x + 2}{x - 3} \quad \text{Ans.}$$

Check, when $x = 4$:

Example:

Answer:

$$\frac{x^2 - 4}{x^2 - 5x + 6} =$$

$$\frac{16 - 4}{16 - 20 + 6} =$$

$$\frac{12}{2} =$$

$$6 \longleftarrow \text{-----} \longrightarrow 6$$

$$\frac{x + 2}{x + 3} =$$

$$\frac{4 + 2}{4 - 3} =$$

EXERCISE 77

Reduce to lowest terms:

1. $\frac{36}{60}$

4. $\frac{28 abc^2}{12 ab^2c}$

2. $\frac{a^2b}{abc}$

5. $\frac{24 a^2x^3}{20 a^2x^4}$

3. $\frac{5 xy^2z}{10 x^3y^2}$

6. $\frac{\pi R^2h}{\pi r^2h}$

7. $\frac{10(p+q)}{18(p+q)}$

11. $\frac{a^2 - b^2}{2a + 2b}$

8. $\frac{3a + 9m}{5(a + 3m)}$

12. $\frac{ax + bx + cx}{ay + by + cy}$

9. $\frac{3a + 3b}{7a + 7b}$

13. $\frac{3a - 6b + 3c}{6a + 12b - 9c}$

10. $\frac{10x - 10y}{10n - 10}$

14. $\frac{4l + 4h}{2lw + 2hw}$

Check each of the following examples, using the values given:

15. $\frac{x^2 + 7x + 10}{3x + 6}$

$x = 1$

16. $\frac{w^2 + 4w}{w^2 + 5w + 4}$

$w = 1$

17. $\frac{2ac + cx}{2ac - cx}$

$a = 2, c = 1, x = 1$

18. $\frac{mx - my}{x^2 - y^2}$

$m = 3, x = 2, y = 1$

19. $\frac{6k - 9}{4k^2 - 9}$

$k = 1$

20. $\frac{y^2 + 8y + 15}{y^2 - 25}$

$y = 1$

21. $\frac{lw}{l^2 + lw}$

$l = 2, w = 1$

22. $\frac{m^2 - 16}{m^2 - 8m + 16}$

$m = 1$

23. $\frac{s^2 - 10s + 21}{s^2 - 6s - 7}$

$s = 2$

24. $\frac{5ac + 10cx}{5bc}$

$a = 1, b = 2, c = 1, x = 2$

$$25. \frac{2 \pi r}{2 \pi r h + 2 \pi r^2} \quad \pi = 3, \quad h = 1, \quad r = 2$$

$$26. \frac{2 x^2 - 5 x + 2}{2 x^2 + 5 x - 3} \quad x = 1$$

$$27. \frac{3 t^2 - 10 t + 3}{t^2 + 3 t - 18} \quad t = 2$$

$$28. \frac{3 y^2 - 6 y z + 9 z^2}{4 a y^2 - 8 a y z + 12 a z^2} \quad y = 1, \quad z = 1, \quad a = 1$$

$$29. \frac{\pi R H + \pi R^2}{\pi H^2 - \pi R^2} \quad \pi = 3, \quad R = 2, \quad H = 1$$

$$30. \frac{4 m^2 - 9 p^2}{4 m^2 - 12 m p + 9 p^2} \quad m = 2, \quad p = 1$$

$$31. \frac{v t + 16 t^2}{3 v + 48 t} \quad v = 30, \quad t = 5$$

32. Reduce to lowest terms the right member of

$$\frac{A}{C} = \frac{\pi R^2}{2 \pi R}$$

33. Can $\frac{x+a}{x+b}$ be reduced to lower terms? Explain.

34. Which of the following fractions have numerator and denominator in factored form:

$$(a) \frac{13}{23}$$

$$(d) \frac{3 m}{3 m + 1}$$

$$(b) \frac{a + 3}{b + 3}$$

$$(e) \frac{a + b(a - b)}{2 a + b}$$

$$(c) \frac{2 \cdot 3}{3(m + 1)}$$

$$(f) \frac{(m + x)(m - x)}{2(m + x)}$$

35. Which of the fractions in Example 34 can be reduced to lowest terms? Reduce them.

$$36. \text{ Reduce to lowest terms } \frac{x^3 - 4 x^2 + 4 x}{3 x - 6}$$

ADDITION AND SUBTRACTION OF FRACTIONS

Oral Exercise

Supply each missing word or number.

$$1. \frac{2}{5} + \frac{3}{5} = ?$$

$$2. \frac{5}{100} + \frac{4}{100} = ?$$

$$3. \frac{5}{7} - \frac{2}{7} = ?$$

4. In adding quarters and eighths one mentally changes them both to —.

5. In subtracting fifths from halves one would change them both to —.

$$6. \frac{3}{8} + \frac{1}{16} = \frac{?}{16}$$

$$10. \frac{c}{m} - \frac{d}{m} = \frac{? - ?}{m}$$

$$7. \frac{1}{3} + \frac{1}{2} = \frac{?}{6} + \frac{?}{6} = ?$$

$$11. \frac{5}{x} + \frac{3}{2x} = \frac{10}{?} + \frac{?}{?} = ?$$

$$8. \frac{1}{3} - \frac{1}{4} = \frac{?}{12} - \frac{?}{12} = ?$$

$$12. \frac{a}{b} + \frac{c}{d} = \frac{?}{bd} + \frac{?}{?} = ?$$

$$9. \frac{p}{q} + \frac{r}{q} = \frac{? + ?}{q}$$

13. What must be true before two fractions can be combined in a single fraction?

In order that fractions may be added or subtracted it is necessary that the fractions have the same denominator. If they have the same denominator the numerators are combined to make the new numerator. If they have different denominators they must first be changed to equivalent fractions with a common denominator. The common denominator must be divisible by each of the denominators.

Example 1: Combine into a single fraction $\frac{3}{16} + \frac{5}{16} - \frac{7}{16}$

Solution: $\frac{3}{16} + \frac{5}{16} - \frac{7}{16} = \frac{3+5-7}{16} = \frac{1}{16}$ Ans.

Example 2: Write as a single fraction $\frac{5a}{x} - \frac{b}{x} + \frac{c}{x}$

Solution: $\frac{5a}{x} - \frac{b}{x} + \frac{c}{x} = \frac{5a-b+c}{x}$ Ans.

Check: Let $a = 1, b = 1, c = 1, x = 2$

<i>Example:</i> $\frac{5a}{x} - \frac{b}{x} + \frac{c}{x} = \frac{5}{2} - \frac{1}{2} + \frac{1}{2}$		<i>Answer:</i> $\frac{5a-b+c}{x} = \frac{5-1+1}{2}$
$= \frac{5}{2}$	$\longleftrightarrow$	$= \frac{5}{2}$

Example 3: Perform the indicated operations: |

$$\frac{3}{4} + \frac{5}{8} - \frac{9}{16}$$

Solution:

$$\frac{3}{4} + \frac{5}{8} - \frac{9}{16} = \frac{12}{16} + \frac{10}{16} - \frac{9}{16} = \frac{12+10-9}{16} = \frac{13}{16} \quad \text{Ans.}$$

Example 4: Combine into one fraction $\frac{1}{ac} + \frac{1}{bc} - \frac{1}{ab}$

Solution:

$$\frac{1}{ac} + \frac{1}{bc} - \frac{1}{ab} = \frac{b}{abc} + \frac{a}{abc} - \frac{c}{abc} = \frac{b+a-c}{abc} \quad \text{Ans.}$$

Check:

Let $a = 2, b = 1, c = 2$

<i>Example:</i> $\frac{1}{ac} + \frac{1}{bc} - \frac{1}{ab} = \frac{1}{4} + \frac{1}{2} - \frac{1}{2}$		<i>Answer:</i> $\frac{b+a-c}{abc} = \frac{1+2-2}{2 \cdot 1 \cdot 2}$
$= \frac{1}{4}$	$\longleftrightarrow$	$= \frac{1}{4}$

Oral Exercise

1. $\frac{3}{8} + \frac{4}{8}$

2. $\frac{22}{7} - \frac{17}{7}$

3. $\frac{3}{x} + \frac{5}{x}$

4. $\frac{11}{2y} - \frac{5}{2y}$

5. $\frac{3}{16} + \frac{7}{16} - \frac{5}{16}$

6. $\frac{7}{2x} - \frac{5}{2x} + \frac{3}{2x}$

7. $\frac{a}{3} - \frac{b}{3} - \frac{c}{3}$

8. $\frac{2r}{ab} - \frac{r}{ab} + \frac{7r}{ab}$

9. $\frac{1}{2} + \frac{1}{3}$

10. $\frac{2}{3} - \frac{1}{2}$

11. $\frac{3}{4} + \frac{2}{3}$

12. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

13. $\frac{1}{2} - \frac{1}{8} - \frac{1}{16}$

14. $\frac{1}{a} + \frac{1}{b}$

15. $\frac{1}{x} + \frac{1}{y}$

16. $\frac{1}{a} - \frac{1}{b}$

17. $\frac{1}{x} - \frac{1}{y}$

18. $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$

19. $1 - \frac{1}{7}$

20. $1 - \frac{3}{5}$

21. $4 - \frac{1}{2}$

22. $3 - \frac{2}{3}$

23. $1 - \frac{1}{a}$

24. $x - \frac{1}{x}$

25. $a - \frac{b}{c}$

26. $\frac{x}{y} + 1$

27. $\frac{m}{n} - 1$

28. $\frac{a}{b} + \frac{b}{c}$

EXERCISE 78

Perform the indicated operations:

$$1. \frac{3}{4} + \frac{5}{16} - \frac{7}{32}$$

$$7. \frac{1}{xy} - \frac{1}{yz} - \frac{1}{xz}$$

$$2. \frac{17}{64} - \frac{3}{32} - \frac{5}{16}$$

$$8. \frac{2a}{3} + \frac{3a}{4} + \frac{5a}{6}$$

$$3. \frac{x}{y} + \frac{y}{z}$$

$$9. \frac{3}{2m} - \frac{4}{3m} + \frac{5}{6m}$$

$$4. \frac{2}{ab} + \frac{5}{ac}$$

$$10. 11 - \frac{5}{6}$$

$$5. \frac{1}{3y} - \frac{1}{3x}$$

$$11. 9 - \frac{3}{4}$$

$$6. \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}$$

$$12. 3b + \frac{5b}{8}; \text{ check when } b = 1$$

Solution:

$$3b + \frac{5b}{8} = \frac{24b}{8} + \frac{5b}{8} = \frac{24b + 5b}{8} = \frac{29b}{8} \quad \text{Ans.}$$

Check:

Example:

Answer:

$$3b + \frac{5b}{8} = 3 + \frac{5}{8}$$

$$\frac{29b}{8} = \frac{29}{8}$$

$$= \frac{29}{8} \longleftrightarrow = \frac{29}{8}$$

Check each of the following examples, using the values given:

$$13. n + \frac{1}{n}$$

$$n = 2$$

$$14. 2a + \frac{5a}{7}$$

$$a = 1$$

$$15. \quad 3x - \frac{5x}{6} \qquad x = 6$$

$$16. \quad 2 + \frac{3}{a} \qquad a = 6$$

$$17. \quad 8 - \frac{7}{3t} \qquad t = 1$$

$$18. \quad a + b + \frac{1}{c} \qquad a = 1, \quad b = 1, \quad c = 2$$

$$19. \quad (a + b) + \frac{1}{c} \qquad a = 2, \quad b = 2, \quad c = 3$$

$$20. \quad b + \frac{br}{100} \qquad b = 1000, \quad r = 5$$

$$21. \quad p + \frac{prt}{100} \qquad p = 500, \quad r = 6, \quad t = 2$$

$$22. \quad p + \frac{prn}{365} \qquad p = 100, \quad r = .05, \quad n = 73$$

$$23. \quad \frac{5}{a+b} + \frac{7}{a+b} \qquad a = 1, \quad b = 1$$

$$24. \quad \frac{3m}{a-b} + \frac{5}{a-b} \qquad m = 1, \quad a = 2, \quad b = 1$$

$$25. \quad \frac{7x}{x-y} - \frac{7y}{x-y} \qquad x = 2, \quad y = 1 \quad \text{Reduce to lowest terms}$$

$$26. \quad \frac{c}{a+b} + \frac{c}{a-b} \qquad c = 3, \quad a = 2, \quad b = 1$$

Solution:

$$\begin{aligned} \frac{c}{a+b} + \frac{c}{a-b} &= \frac{(a-b)c}{(a-b)(a+b)} + \frac{(a+b)c}{(a+b)(a-b)} \\ &= \frac{ac - bc}{(a-b)(a+b)} + \frac{ac + bc}{(a+b)(a-b)} \\ &= \frac{ac - bc + ac + bc}{(a+b)(a-b)} \\ &= \frac{2ac}{(a+b)(a-b)} \quad \text{Ans.} \end{aligned}$$

Check:

Example:

$$\frac{c}{a+b} + \frac{c}{a-b} = \frac{3}{2+1} + \frac{3}{2-1}$$

$$= 1+3$$

Answer:

$$\frac{2ac}{(a+b)(a-b)} = \frac{2 \cdot 2 \cdot 3}{(2+1)(2-1)}$$

$$= \frac{12}{3}$$

$$= 4 \longleftarrow \longrightarrow = 4$$

In many examples such as 26, the common denominator to be used will be the product of the given denominators. In others it will be necessary to factor each denominator carefully and write the product of the different factors as the common denominator; no factor need be used more times than it appears in any one denominator. For example; to add $\frac{1}{a^2}$

and $\frac{1}{ab}$ the lowest common denominator is a^2b ; to add $\frac{1}{8a}$ and $\frac{1}{4x}$ the lowest common denominator (l. c. d.) is $8ax$.

Before deciding what the l. c. d. is, remember that you must first factor the denominators.

Be sure to reduce the answers to lowest terms.

27. In adding the fractions $\frac{3}{2a^2b}$ and $\frac{5}{4ab^2}$ what will be the lowest common denominator?

28. Find what l. c. d. you will use in adding the fractions $\frac{7a}{3a+3b}$ and $\frac{5c}{6a+6b}$.

29. The denominators of two fractions are $r+s$ and r^2-s^2 respectively. Find the l. c. d.

30. Find the l. c. d. if the denominators are $6(m^2-n^2)$ and $3(m^2-3mn+2n^2)$.

Perform the following additions and subtractions using in each case the lowest common denominator, and reducing

your answer to lowest terms. Check each example, assigning whatever values you please to the letters. If any check number makes a denominator zero, change to a different number.

$$31. \frac{5x-3}{2} + \frac{x+1}{5}$$

$$35. \frac{3}{x+3} + \frac{5}{x-4}$$

$$32. \frac{3n+1}{4} + \frac{3n-1}{2}$$

$$36. \frac{2}{y-5} + \frac{4}{y+3}$$

$$33. \frac{y}{2} + \frac{y-1}{3} + \frac{y-2}{4}$$

$$37. \frac{7}{z+6} + \frac{2}{z-3}$$

$$34. \frac{1}{m+n} + \frac{1}{m-n}$$

$$38. \frac{m+n}{5} - \frac{m-n}{10}$$

Solution:

$$\begin{aligned} \frac{m+n}{5} - \frac{m-n}{10} &= \frac{2(m+n)}{10} - \frac{m-n}{10} \\ &= \frac{(2m+2n) - (m-n)}{10} \\ &= \frac{2m+2n-m+n}{10} \\ &= \frac{m+3n}{10} \quad \text{Ans.} \end{aligned}$$

Check: when $m = 2, n = 1$

Example:

$$\frac{m+n}{5} - \frac{m-n}{10} =$$

$$\frac{2+1}{5} - \frac{2-1}{10} =$$

$$\frac{3}{5} - \frac{1}{10} = \frac{5}{10}$$

Answer:

$$\frac{m+3n}{10} =$$

$$\frac{2+3}{10} = \frac{5}{10}$$

$$39. \frac{2m+3}{4} - \frac{3m+1}{3} + \frac{m-5}{12}$$

40. $\frac{3t-6}{2} - \frac{2t-3}{3}$ 45. $\frac{2x}{y^2-x^2} + \frac{1}{y+x} + \frac{2}{y-x}$
41. $\frac{5}{2r+s} - \frac{3}{2r-s}$ 46. $\frac{3}{(x-2)(x-3)} + \frac{3}{x-3}$
42. $\frac{7}{p-q} - \frac{3}{2p+q}$ 47. $\frac{3}{x^2-7x+10} + \frac{1}{x-2}$
43. $\frac{b}{a^2-b^2} + \frac{1}{a+b}$ 48. $\frac{1}{t-s} - \frac{s}{t^2-s^2}$
44. $\frac{1}{x+y} + \frac{y}{x^2-y^2}$ 49. $\frac{c+d}{c-d} - \frac{2cd}{c^2-d^2}$
50. $\frac{3}{x-8} - \frac{6}{x^2-7x-8}$
51. $(a+b) + \frac{b^2}{a-b}$ $\left[\text{Write } (a+b) \text{ as } \frac{a+b}{1} \right]$
52. $(c-d) + \frac{d^2}{c+d}$ 55. $\frac{x+2}{x-3} - \frac{x-3}{x+5}$
53. $1 - \frac{b^2}{a^2-b^2}$ 56. $\frac{y+6}{y-5} - \frac{y+2}{y+3}$
54. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$ 57. $\frac{2m+5}{m^2+5m-14} - \frac{1}{m+7}$
58. $\frac{2}{t-3} + \frac{3}{t+4} + \frac{5-4t}{t^2+t-12}$
59. $\frac{5}{c-2} + \frac{4}{c+9} - \frac{8c+28}{c^2+7c-18}$
60. $\frac{3x}{x-5} - \frac{x^2+x}{x^2-16x+55} + \frac{2x}{x-11}$
61. $\frac{5k}{2k+7} + \frac{12k}{4k^2-49} - \frac{3}{2k-7}$
62. $\frac{12y^2+2y-10}{3y^2-10y+3} - \frac{3}{3y-1} - \frac{2y+1}{y-3}$
63. $\frac{5}{x} - \frac{3x}{x^2} - \frac{1}{x+3}$ 64. $\frac{2}{s-2} - \frac{1}{s-3} - \frac{1}{s-4}$

MULTIPLICATION OF FRACTIONS

Algebraic fractions are multiplied just as are fractions in arithmetic. The numerators are multiplied together to form the new numerator and the denominators are multiplied to form the new denominator. The resulting fraction is then reduced to lowest terms. The work is made much easier if factors common to a numerator and any denominator are first cancelled.

Example 1: Multiply $\frac{15}{16}$ by $\frac{8}{25}$ by $\frac{5}{9}$.

Solution:

$$\frac{\overset{3}{\cancel{15}}}{\underset{2}{\cancel{16}}} \cdot \frac{\overset{2}{\cancel{8}}}{\underset{5}{\cancel{25}}} \cdot \frac{\overset{5}{\cancel{9}}}{3} = \frac{1}{6}$$

Example 2: Perform the indicated operation:

$$\frac{3a + 3b}{5a - 5b} \cdot \frac{4a^2 - 6ab + 2b^2}{6a^2 - 6b^2} \cdot \frac{10a - 10b}{2ac - bc}$$

Solution:

$$\frac{3a + 3b}{5a - 5b} \cdot \frac{4a^2 - 6ab + 2b^2}{6a^2 - 6b^2} \cdot \frac{10a - 10b}{2ac - bc}$$

$$\frac{\cancel{3}(a+b)}{\cancel{5}(a-b)} \cdot \frac{\cancel{2}(2a-b)(a-b)}{\cancel{6}(a+b)(a-b)} \cdot \frac{\overset{2}{\cancel{10}(a-b)}}{c(\cancel{2}a-b)} = \frac{2}{c} \text{ Ans.}$$

Check:

Let $a = 2$, $b = 1$, $c = 1$

Example:

Answer:

$$\frac{3a + 3b}{5a - 5b} \cdot \frac{4a^2 - 6ab + 2b^2}{6a^2 - 6b^2} \cdot \frac{10a - 10b}{2ac - bc} = \frac{2}{c}$$

$$\frac{6 + 3}{10 - 5} \cdot \frac{16 - 12 + 2}{24 - 6} \cdot \frac{20 - 10}{4 - 1} = \frac{2}{1}$$

$$\frac{9}{5} \cdot \frac{6}{18} \cdot \frac{\overset{2}{\cancel{10}}}{3} = 2 \longleftrightarrow 2$$

Oral Exercise

Multiply:

- | | |
|-------------------------------------|--|
| 1. $\frac{2}{3}$ by $\frac{5}{7}$ | 11. $\frac{m}{n}$ by p |
| 2. $\frac{3}{5}$ by $\frac{15}{16}$ | 12. $\frac{a}{c^2}$ by c |
| 3. $\frac{y}{z}$ by $\frac{r}{s}$ | 13. $\frac{a}{c}$ by x |
| 4. $\frac{m}{n}$ by $\frac{p}{q}$ | 14. $\frac{a}{b}$ by $\frac{a'}{b'}$ |
| 5. $\frac{7}{1}$ by $\frac{2}{3}$ | 15. $\frac{b}{b'}$ by $\frac{a}{a'}$ |
| 6. 5 by $\frac{3}{4}$ | 16. $\frac{a}{b}$ by $\frac{b^2}{a^2}$ |
| 7. 3 by $\frac{1}{6}$ | 17. $\frac{ax}{b}$ by $\frac{m}{ax}$ |
| 8. a by $\frac{x}{y}$ | 18. $\frac{3(p+q)}{c(m+n)}$ by $\frac{m+n}{p+q}$ |
| 9. c by $\frac{a}{b}$ | 19. $\frac{x+y}{3(x-y)}$ by $\frac{c(x-y)}{x+y}$ |
| 10. $\frac{22}{7}$ by 14 | 20. $\frac{a(m+n)}{c(m-n)}$ by $\frac{c}{m+n}$ |

EXERCISE 79

Perform the indicated multiplications and check by numerical substitution:

- | | |
|---|-------------------------------------|
| 1. $\frac{7h}{2R^2} \cdot \frac{22Rh}{7h^2}$ | 4. $3abc \cdot \frac{xy}{6a^2bc}$ |
| 2. $\frac{r^2s}{mn} \cdot \frac{m^2n^2}{r^3s^2}$ | 5. $6a^2b^2 \cdot \frac{3mn}{18ab}$ |
| 3. $\frac{xyz}{x^2z} \cdot \frac{x^3z^3}{y^2z^2}$ | 6. $(x-5) \cdot \frac{5x}{3(x-5)}$ |

7. $(2x - 6) \cdot \frac{7a}{14(x - 3)}$
8. $(2r + 3s) \cdot \frac{t^2}{(2r + 3s)(t - 1)}$
9. $\frac{2a - 2b}{3m + 3n} \cdot \frac{6m + 6n}{a^2 - b^2}$
10. $\frac{p^2 + 3pq + 2q^2}{p^2 - 2pq - 3q^2} \cdot \frac{2p - 6q}{3p + 6q}$
11. $\frac{2t^2 + 5ts + 2s^2}{2t^2 - st - s^2} \cdot \frac{t^2 - s^2}{4s^2 + 4ts + t^2}$
12. $\frac{x^2 - x - 6}{x^2 - 8x + 15} \cdot \frac{x^2 - 3x - 10}{x^2 - 5x - 14}$
13. $\frac{3a^2b + 6ab^2}{2c^2d + 4cd^2} \cdot \frac{3c^2 + 6cd}{2ab + 4b^2}$
14. $\frac{4m^2 - 9n^2}{6m^2 - 9mn} \cdot \frac{6mn}{4mn + 6n^2}$
15. $\frac{2p^3 - 2p^2q - 4pq^2}{5p^2q - 5q^3} \cdot \frac{30pq^2 - 15q^3}{12p^3 - 30p^2q + 12pq^2}$
16. $\frac{5x^2 - 14xy - 3y^2}{25x^2y - y^3} \cdot \frac{10x^3 + 3x^2y - xy^2}{2x^2 - 5xy - 3y^2}$
17. $(x^2 - 4x - 21) \cdot \frac{2xy}{3x^2 - 21x}$
18. $\frac{6rs + 2rs^2}{6rs^2 - 9s^3} \cdot \frac{3s^3}{9r^2 - s^2} \cdot \frac{6r^2 - 11rs + 3s^2}{2r^2 - 2rs}$
19. $\frac{(x - 2)^3}{x^2 - 3x - 10} \cdot \frac{x + 2}{(x - 2)^2}$
20. $(3x^3 - 75x) \cdot \frac{2x + 1}{3x^2 + 15x}$
21. $(2a - b)(a^2 - b^2) \cdot \frac{3c}{2a^2b + ab^2 - b^3}$
22. $\left(\frac{a}{b} + \frac{b}{a}\right) \cdot \frac{a^2b^2}{a^2 + b^2}$

$$23. \left(1 - \frac{y}{x}\right) \cdot \frac{x}{x^2 - y^2}$$

$$24. \left(\frac{m^2}{n^2} - 9\right) \cdot \left(\frac{n}{m - 3n}\right)$$

$$25. \left(\frac{1}{q^2} - \frac{5}{pq} + \frac{6}{p^2}\right) \cdot \frac{pq}{p - 3q}$$

$$26. \left(2 - \frac{5}{t} + \frac{2}{t^2}\right) \cdot \left(1 + \frac{4}{t^2 - 4}\right)$$

$$27. \left(r - 4s - \frac{10s^2}{r - s}\right) \cdot \left(2r + s + \frac{12rs + 3s^2}{r - 6s}\right)$$

28. Multiply each term of the following expression by 30. Collect like terms in the product.

$$\frac{2n - 4}{5} + \frac{3n + 2}{10} + \frac{7n - 14}{15} \quad \text{Ans. } 35n - 46$$

29. Multiply each fraction by $(x - 5)(x + 2)$ and then collect terms:

$$\frac{2x}{x - 5} + \frac{3x}{x + 2} + \frac{4 - 5x^2}{x^2 - 3x - 10}$$

30. Multiply the following expressions by the l. c. d. of the fractions and simplify:

$$\frac{2c + 3}{3c} + \frac{2c^2 - 7}{4c^2} - \frac{7c - 6}{6c}$$

31. Multiply each term in the following equation by $(3t - 7)(3t + 7)$. Simplify each member as much as possible.

$$\frac{2t - 3}{3t - 7} - \frac{4t + 5}{9t^2 - 49} = \frac{3t + 2}{3t + 7}$$

32. Change the following equation into one without fractions by multiplying each member by the same quantity:

$$\frac{2x + 1}{x + 9} - \frac{3x - 10}{x - 3} = \frac{5 - x^2}{x^2 + 6x - 18}$$

DIVISION OF FRACTIONS

To divide one algebraic fraction by another, the same method is used as in arithmetic. Multiply the dividend by the reciprocal of the divisor.

Example 1:

Divide $\frac{x^2 + 9x - 22}{2x + 22}$ by $\frac{x^2 + 7x - 18}{x^2 + 9x}$

Solution:

$$\begin{aligned} \frac{x^2 + 9x - 22}{2x + 22} \div \frac{x^2 + 7x - 18}{x^2 + 9x} &= \frac{x^2 + 9x - 22}{2x + 22} \cdot \frac{x^2 + 9x}{x^2 + 7x - 18} \\ &= \frac{\cancel{(x+11)} \cancel{(x-2)}}{2\cancel{(x+11)}} \cdot \frac{x\cancel{(x+9)}}{\cancel{(x+9)} \cancel{(x-2)}} \\ &= \frac{x}{2} \text{ Ans.} \end{aligned}$$

Check:

Let $x = 1$

Example:

Answer:

$$\begin{array}{l|l} \frac{x^2 + 9x - 22}{2x + 22} \div \frac{x^2 + 7x - 18}{x^2 + 9x} & \frac{x}{2} = \frac{1}{2} \\ \frac{1 + 9 - 22}{2 + 22} \div \frac{1 + 7 - 18}{1 + 9} & \\ \frac{-12}{24} \div \frac{-10}{10} = \frac{1}{2} & \longleftrightarrow = \frac{1}{2} \end{array}$$

EXERCISE 80

Carry out the indicated operations in the following examples:

1. $\frac{2}{3} \div \frac{3}{4}$

4. $\frac{15}{16} \div \frac{5}{8}$

2. $\frac{5}{6} \div \frac{1}{2}$

5. $10 \div \frac{2}{5}$

3. $\frac{7}{8} \div 2$

6. $2\frac{1}{2} \div \frac{1}{4}$

7. $\frac{x}{y} \div \frac{m}{n}$

8. $\frac{a}{b} \div \frac{c}{d}$

9. $\frac{m^2}{n^2} \div \frac{m}{n}$

10. $\frac{a}{x} \div \frac{1}{x}$

11. $\frac{a}{x} \div a$

12. $b \div \frac{b}{c}$

13. $\frac{ab}{cd} \div \frac{a}{c}$

14. $\frac{p^2q}{r^2s} \div \frac{pq^2}{rs^2}$

15. $\frac{4y^2z}{3a^2b} \div \frac{2yz^2}{3ab^2}$

16. $\frac{12m^4n}{15cd^3} \div \frac{4m^2n^2}{5c^2d}$

17. $\frac{2ab}{5rs} \div \frac{6a^2b^2}{25r^2s^2}$

18. $\frac{x+y}{2a+2b} \div \frac{x-y}{3a+3b}$

Check Examples 19–28, assigning to the letters such values as you please. If any check number makes a factor zero, choose a different number.

19. $\frac{a+b}{x^2-y^2} \div \frac{2}{3(x+y)}$

22. $\frac{t-1}{t} \div \frac{t^2-1}{t^2}$

20. $\frac{(a+b)^2}{m-n} \div \frac{a^2-b^2}{m+n}$

23. $\frac{x^2-5x+6}{x^2-6x-7} \div \frac{x-3}{x-7}$

21. $\frac{m^2-4n^2}{3} \div (m+2n)$

24. $\frac{p+7}{p+3} \div \frac{p^2+2p-35}{p^2-p-12}$

25. $\frac{a^2+8ab-33b^2}{a^2-14ab+33b^2} \div \frac{a^2-121b^2}{a^2-6ab-55b^2}$

26. $\frac{6x^2+5xy-21y^2}{21x^2-5xy-6y^2} \div \frac{3x^2+7xy}{7xy+3y^2}$

27. $\frac{3m^2+mn-4n^2}{3m^2+7mn+4n^2} \div \frac{m^2-n^2}{5m+5n}$

28. $\frac{4r^2-25s^2}{5r^2+4rs-s^2} \div \frac{2r^2-3rs-5s^2}{5r^2-6rs+s^2}$

29. $\left(\frac{a}{m} + \frac{b}{m}\right) \div \left(\frac{a}{n} + \frac{b}{n}\right)$

30. $\left(\frac{a}{x} + \frac{b}{x}\right) \div \left(\frac{a^2}{x^2} - \frac{b^2}{x^2}\right)$ 32. $\left(a - \frac{b}{c}\right) \div \left(a + \frac{b}{c}\right)$
31. $\left(1 + \frac{1}{m}\right) \div \left(1 - \frac{1}{m}\right)$ 33. $\left(\frac{1}{r} + \frac{1}{s}\right) \div \left(\frac{1}{r^2} - \frac{1}{s^2}\right)$
34. $\left(\frac{2}{h^2} - \frac{5}{h} + 2\right) \div \left(\frac{2}{h} - 1\right)$
35. $\left(\frac{t^2}{s^2} - \frac{8t}{s} + 15\right) \div \left(\frac{t}{s} - 3\right)$
36. $\left(y + 3 - \frac{6}{y-2}\right) \div \left(y - 5 + \frac{10y-23}{y-3}\right)$

37. Fill in the blanks: (a) Multiplying a quantity by $\frac{1}{2}$ has the same effect as dividing it by —; (b) Dividing a quantity by 10 has the same effect as multiplying it by —.

38. By what would the quantity $\frac{6x}{x+5}$ have to be multiplied to get $6x$?

39. By what would $\frac{a}{b}$ have to be divided to get $\frac{b}{a}$? Check.

40. Divide $\frac{N}{D}$ by the fraction obtained by inverting it.

41. Write any algebraic fraction. Invert it and multiply by the original fraction. What is the product?

42. What is the product of any fraction and its reciprocal?

COMPLEX FRACTIONS

The expression $\frac{a}{b} \div \frac{c}{d}$ may be written in the form $\frac{\frac{a}{b}}{\frac{c}{d}}$.

This form is called a complex fraction. The easiest way to simplify most complex fractions is to make use of the fundamental principle of fractions. Multiply both numerator and denominator of the complex fraction by some number

EXERCISE 81

Express the following complex fractions as simple fractions.
Check all algebraic fractions by numerical substitution.

$$1. \frac{\frac{a}{m}}{\frac{c}{m}}$$

$$2. \frac{\frac{1}{x}}{\frac{a}{x}}$$

$$3. \frac{1 - \frac{1}{a}}{1 + \frac{1}{a}}$$

$$4. \frac{a + \frac{b}{c}}{a - \frac{b}{c}}$$

$$5. \frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$$

$$6. \frac{\frac{2}{3} + \frac{3}{4}}{1 - \frac{5}{12}}$$

$$7. \frac{\frac{7}{8} - \frac{3}{4}}{\frac{5}{8} - \frac{3}{16}}$$

$$8. \frac{\frac{m}{n} + \frac{n}{m}}{\frac{m}{n} - \frac{n}{m}}$$

$$9. \frac{\frac{1}{a} + \frac{1}{a^2}}{\frac{1}{a} - \frac{1}{a^2}}$$

$$10. \frac{\frac{2b}{c} - \frac{b}{c^2}}{\frac{2b}{c} + \frac{b}{c^2}}$$

$$11. \frac{\frac{x}{4} - \frac{1}{x}}{\frac{x}{4} + \frac{5}{4} + \frac{3}{2x}}$$

$$12. \frac{\frac{a}{b}}{\frac{x}{y}}$$

$$13. \frac{\frac{3}{m}}{\frac{m}{3}}$$

14. Find the value of $\frac{a+b}{a-b}$ if $a = \frac{5}{6}$ and $b = \frac{2}{3}$.

15. Find the value of $\frac{m+n}{1-mn}$ if $m = \frac{1}{2}$ and $n = \frac{1}{3}$.

16. Find the value of $\frac{m-n}{1+mn}$ if $m = \frac{3}{5}$ and $n = \frac{1}{3}$.

17. Find the value of $\frac{x^2 - y^2}{x^2 + xy + y^2}$ if $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

18. What is the value of R in the formula

$$R = \frac{r_1 r_2}{r_1 + r_2} \text{ when } r_1 = \frac{3}{4} \text{ and } r_2 = \frac{2}{3}?$$

19. Find the value of R in the formula

$$R = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3} \text{ when } r_1 = \frac{1}{2}, r_2 = \frac{1}{3}, r_3 = \frac{1}{4}.$$

20. Show that this equation is true if $x = \frac{1}{4}$

$$\frac{x+1}{3-2x} = \frac{3x}{2x+1}$$

21. Investigate whether the preceding equation is true if $x = \frac{1}{2}$.

22. What is the value of $\frac{2a+1}{1-a^2}$ if $a = 0.1$?

CHAPTER X

FRACTIONS IN EQUATIONS — FORMULAS

In solving equations involving fractions we multiply each member of the equation by the same number, so choosing the number that the fractions disappear. The smallest number which we can make use of in this way is the l. c. d. of the fractions. For example, in solving the equation $\frac{2x}{3} = \frac{5}{6} + \frac{x}{4}$ we multiply each member of the equation by 12, which is the l. c. d. of the fractions. We obtain $8x = 10 + 3x$, an equation without fractions, from which we easily obtain the value of x .

Oral Exercise

Solve the following equations and check:

1. $\frac{x}{2} = 3$

2. $\frac{y}{4} = 5$

3. $\frac{2x}{5} = 8$

4. $\frac{3y}{7} = 6$

5. $\frac{m}{2} - 3 = 0$

6. $\frac{r}{7} - 8 = 0$

7. $\frac{2y}{3} - 6 = 0$

8. $\frac{3m}{4} - 3 = 0$

9. $\frac{x}{5} + 3 = 0$

10. $\frac{x}{2} + \frac{x}{3} = 5$

11. $\frac{t}{4} + 1 = 3$

12. $\frac{s}{3} - 1 = 4$

13. $\frac{4}{x} = 2$

14. $\frac{6}{y} = 3$

15. $2 - \frac{1}{y} = 0$

18. $5 = \frac{10}{3x}$

16. $3 - \frac{1}{R} = 0$

19. $\frac{3}{y} + 5 = 0$

17. $\frac{3}{2x} = 1$

20. $\frac{2}{c} - 1 = 1$

21. $\frac{1}{c} + \frac{2}{c} = 3$

Many equations that are somewhat more difficult are solved in just the same way. Keep in mind that the number by which you are going to multiply each member of the equation is the l. c. d. of the fractions in the equation.

Example 1: Solve for x the equation: $\frac{3x}{4} + \frac{7}{8} = \frac{5x}{6} + \frac{7}{12}$ and check.

Solution: $\frac{3x}{4} + \frac{7}{8} = \frac{5x}{6} + \frac{7}{12}$

The l. c. d. is 24; multiplying each member of the equation by 24,

$$\begin{aligned} \cancel{24} \cdot \frac{3x}{4} + \cancel{24} \cdot \frac{7}{8} &= \cancel{24} \cdot \frac{5x}{6} + \cancel{24} \cdot \frac{7}{12} \\ 18x + 21 &= 20x + 14 \\ -2x &= -7 \\ x &= \frac{7}{2} \end{aligned}$$

Check:

Left member		Right member
$\frac{3x}{4} + \frac{7}{8} = \frac{3 \cdot \frac{7}{2}}{4} + \frac{7}{8}$		$\frac{5x}{6} + \frac{7}{12} = \frac{5 \cdot \frac{7}{2}}{6} + \frac{7}{12}$
$= \frac{21}{8} + \frac{7}{8}$		$= \frac{35}{12} + \frac{7}{12}$
$= \frac{7}{2}$	$\longleftrightarrow$	$= \frac{7}{2}$

Example 2: Solve and check: $\frac{t+2}{t+4} - \frac{t-1}{t-4} = \frac{2t-18}{t^2-16}$

Solution: $\frac{t+2}{t+4} - \frac{t-1}{t-4} = \frac{2t-18}{t^2-16}$

Factor $t^2 - 16$

$$\frac{t+2}{t+4} - \frac{t-1}{t-4} = \frac{2t-18}{(t+4)(t-4)}$$

The l. c. d. is $(t+4)(t-4)$; multiply each member by it

$$(t+4)(t-4) \cdot \frac{t+2}{t+4} - (t+4)(t-4) \cdot \frac{t-1}{t-4}$$

$$= (t+4)(t-4) \cdot \frac{2t-18}{(t+4)(t-4)}$$

$$(t-4)(t+2) - (t+4)(t-1) = 2t-18$$

$$(t^2 - 2t - 8) - (t^2 + 3t - 4) = 2t - 18$$

$$t^2 - 2t - 8 - t^2 - 3t + 4 = 2t - 18$$

$$t^2 - t^2 - 2t - 3t - 2t = 8 - 4 - 18$$

$$-7t = -14$$

$$t = 2$$

Check:

Left member	Right member
$\frac{t+2}{t+4} - \frac{t-1}{t-4} = \frac{2+2}{2+4} - \frac{2-1}{2-4}$	$\frac{2t-18}{t^2-16} = \frac{2(2)-18}{(2)^2-16}$
$= \frac{4}{6} - \frac{1}{-2}$	$= \frac{4-18}{4-16}$
$= \frac{4}{6} + \frac{1}{2}$	$= \frac{-14}{-12}$
$= \frac{7}{6}$	$= \frac{7}{6}$

EXERCISE 82

Solve the following equations and check:

1. $\frac{x}{4} + 11 = 15$
2. $\frac{2x}{3} - 5 = 7$
3. $\frac{7y}{8} + \frac{1}{4} = 2$
4. $\frac{3t}{5} - 7 = -4$
5. $\frac{2}{3} + \frac{11x}{6} = 8$
6. $\frac{n}{5} + \frac{n}{6} = 11$
7. $\frac{3z}{4} - 1 = \frac{2z}{3}$
8. $\frac{7m}{2} + m = 18$
9. $\frac{5S}{3} - \frac{2}{9} = S$
10. $\frac{2}{7} - \frac{9x}{21} = -\frac{x}{35}$
11. $\frac{n}{2} + \frac{n}{3} + \frac{n}{5} = 62$
12. $\frac{n}{2} - \frac{n}{3} = \frac{n}{4} - \frac{n}{5} + 14$
13. $\frac{x}{7} + \frac{x+1}{3} + \frac{x+2}{4} = 11$
14. $\frac{120 - S}{S} = 7$
15. $\frac{x+8}{3} + \frac{x}{4} = 19$
16. $\frac{47 - S}{S} = 1 + \frac{9}{S}$
17. $\frac{x-7}{3} + \frac{3x-2}{4} = 8$
18. $\frac{3x-1}{4} - 1 = \frac{2x-1}{5}$
19. $n - \frac{n+1}{2} = 3$
20. $10 - \frac{2x+3}{7} = 5x - 1$
21. $\frac{2y+1}{5} - \frac{y-2}{15} = 6$
22. $\frac{4z-3}{3} - \frac{z+8}{3} = 5$
23. $5 - \frac{S+1}{3} = S - \frac{S-1}{2}$
24. $\frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{1}{x}$
25. $\frac{1}{x} = \frac{1}{6} - \frac{1}{10}$
26. $\frac{14}{x} + \frac{21}{x} = 5$
27. $\frac{5}{2y} + \frac{7}{3y} = \frac{29}{6}$
28. $\frac{7n-5}{3n} - 1 = \frac{5}{n}$
29. $7 - \frac{4t-8}{3t} = \frac{27t+24}{5t}$
30. $\frac{5}{x+3} = 2$

$$31. \frac{3K+1}{K-2} = 4$$

$$36. \frac{2z-7}{3z+1} = \frac{3}{16}$$

$$32. \frac{12}{n-2} + \frac{4}{3(n-2)} = \frac{40}{9}$$

$$37. \frac{4}{x+1} = \frac{2}{x-2}$$

$$33. \frac{x}{x-7} = \frac{9}{2}$$

$$38. \frac{3}{t+21} - \frac{2}{t-4} = 0$$

$$34. \frac{4+c}{7+c} = \frac{3}{4}$$

$$39. \frac{10}{n-1} - \frac{9}{n-11} = 0$$

$$35. \frac{2n-5}{n+6} = \frac{1}{9}$$

$$40. \frac{b-4}{2b-5} = \frac{b-11}{2b-13}$$

$$41. \frac{3z-4}{z+6} - \frac{3z-27}{z+13} = 0$$

$$42. \frac{1}{x-3} + \frac{4}{x+3} = \frac{16}{x^2-9}$$

$$43. \frac{10}{S+4} - \frac{1}{S-4} = \frac{10}{S^2-16}$$

$$44. \frac{y+4}{2y+3} - \frac{y-1}{2y-3} = \frac{3}{4y^2-9}$$

$$45. \frac{3}{m+2} + \frac{2}{m+3} = \frac{33}{(m+2)(m+3)}$$

$$46. \frac{7x-11}{x^2-7x-8} - \frac{7}{x-8} = \frac{3}{x+1}$$

$$47. \frac{4x-6}{x^2-3x-10} - \frac{3}{x-5} = \frac{5}{x+2}$$

$$48. \frac{b+4}{b-7} - \frac{b-3}{b+4} = \frac{16b-9}{b^2-3b-28}$$

$$49. \frac{z+9}{z-5} - \frac{4z+7}{z^2-12z+35} = \frac{z+4}{z-7}$$

$$50. \frac{x}{2x+3} + \frac{x+1}{3x-2} = \frac{5x^2+24}{6x^2+5x-6}$$

$$51. 1 - \frac{1}{n-1} = \frac{n^2-7}{(n-1)^2}$$

$$52. \frac{5}{z} + \frac{z-3}{z^2-5z} + \frac{1}{z-5} = 0$$

In solving the following equations, first multiply each member by 10 or by 100:

$$53. 2.3x = 46$$

$$54. p + .07p = \$535$$

$$55. c - 0.4c = \$675$$

$$56. 2R + 3.14R = 10 \text{ (to three figures)}$$

LITERAL EQUATIONS AND FORMULAS

In a literal equation one or more of the known quantities are represented by letters. It is important at each step of a solution to keep in mind which of the letters is the one that represents the unknown quantity. Literal equations are solved by the same principles and methods as are other equations. Some easy literal equations have already been solved in previous chapters.

Oral Exercise

Solve each of the following equations for x :

$$1. 3x = 6c$$

$$2. 10x = d$$

$$3. ax = b$$

$$4. \frac{x}{5} = p$$

$$5. \frac{x}{m} = n$$

$$6. x - 5a = 0$$

$$7. x + 5 = a$$

$$8. \frac{2x}{7} = b$$

$$9. \frac{px}{q} = r$$

$$10. \frac{1}{2}x - a = 0$$

$$11. \frac{x}{b} + c = 0$$

$$12. 8x - k = 0$$

$$13. 9x - 7w = 25$$

$$14. ax + b = c$$

$$15. 2x = a + b$$

$$16. (r + s)x = m$$

A formula, as we have seen, represents a statement which has been changed or translated from a sentence expressed in words to an equation in the shorthand of algebra. The statement that "the area of a rectangle is equal to the product of its length and its width" was changed to the formula $a = lw$. Just as the word *area* is the subject of the sentence, so we may think of a as the subject of the formula. By dividing each member of the formula $lw = a$ by w , we get $l = \frac{a}{w}$, and l may now be considered as the subject of the formula. To *change the subject of the formula* $v = cn$ to n means the same as to solve the formula $v = cn$ for n .

Example: Change the subject of the formula $v = cn$ to n .

Solution: $v = cn$

Reversing the equation, $cn = v$

Dividing each member by c , $n = \frac{v}{c}$

Oral Exercise

In each of the following formulas change the subject of the formula to the letter indicated. Express each of the formulas as a word sentence. The page reference, at the right, tells where the formula was first used in this book.

- | | | |
|--------------------|-----|------------|
| 1. $p = 4S$ | (S) | (page 11) |
| 2. $A = lw$ | (w) | (page 7) |
| 3. $d = rt$ | (t) | (page 24) |
| 4. $p = br$ | (r) | (page 240) |
| 5. $C = \pi D$ | (D) | (page 38) |
| 6. $i = prt$ | (p) | (page 31) |
| 7. $C = 2\pi R$ | (R) | (page 38) |
| 8. $v = lwh$ | (w) | (page 25) |
| 9. $v = \pi R^2 h$ | (h) | (page 173) |
| 10. $L = 2\pi rh$ | (r) | (page 166) |

The longer and more complicated formulas are handled just as were the equations on pages 217-218.

Example 1: Solve the formula $p = 4S - 2t$ for t .

Solution: $p = 4S - 2t$.

Transposing, $2t = 4S - p$

Dividing by 2, $t = \frac{4S - p}{2}$ Ans.

Check:

Left member

$$p$$

Right member

$$4S - 2t$$

$$4S - 2 \frac{(4S - p)}{2}$$

$$4S - 4S + p$$

$$p \longleftrightarrow p$$

Example 2: Change the subject of the formula $E = \frac{C}{R + nr}$ to r .

Solution: $E = \frac{C}{R + nr}$

Multiplying each member by $R + nr$,

$$(R + nr) E = (R + nr) \cdot \frac{C}{R + nr}$$

$$ER + Enr = C$$

$$Enr = C - ER$$

$$r = \frac{C - ER}{nE} \text{ Ans.}$$

EXERCISE 83

Change the subject of each formula to the letter indicated:

1. $S = C + g$ (g) (page 28)
2. $a = p + i$ (i) (page 242)
3. $p = a + b + c$ (b) (page 25)
4. $S = c - l$ (l) (page 29)

5. $d = D - 2t$ (D) (page 29)
6. $S = L + 2B$ (L) (page 26)
7. $s = b - br$ (r) (page 162)
8. $p = 2l + 2w$ (w) (page 12)
9. $A = \frac{1}{2}bh$ (h) (page 44)
10. $V = \frac{1}{3}Bh$ (B) (page 158)
11. $i = \frac{prn}{360}$ (n) (page 31)
12. $\frac{W}{w} = \frac{l}{L}$ (L) (page 248)

13. Using your answer to Example 8, find the width of a rectangle whose perimeter is 42" and whose length is 12".

14. From the result of Example 9 find the altitude of a triangle whose area is 72.68 square inches and whose base is 18.4 inches.

15. The formula for the total surface of a right cylinder is $T = 2\pi r(h + r)$. (See page 182.) Find the value of h in terms of T and r , — that is, solve the formula for h .

16. Find the height of a right cylinder whose total surface is 3520 square feet, if the radius of its base is 14 feet. Take $\pi = 3\frac{1}{7}$ and use your answer to Example 15.

FACTORING AS AN AID IN THE SOLUTION OF EQUATIONS

In many formulas, we find two or more terms each involving the unknown quantity. When such terms cannot be combined in one, the unknown letter must be taken out as a common factor.

Example: Solve for f the formula $mf = mc - cf$

Solution: $mf = mc - cf$

Transposing, $mf + cf = mc$

Factoring the left member, $f(m + c) = mc$

Dividing each member by $m + c$, $f = \frac{mc}{m + c}$ Ans.

EXERCISE 84

Solve each of the following formulas for the letter indicated:

1. $ax + bx = c$ (x)

2. $A = 2lh + 2wh$ (h) (page 180)

3. $S = b + br$ (b) (page 242)

4. $S = b - br$ (b) (page 162)

5. $rS - S = rl - a$ (S)

6. $a + ar + ar^2 = S$ (a)

7. $V = \pi R^2 h - \pi r^2 h$ (h) (page 173)

8. $S = b - b \cdot \frac{R}{100}$ (b)

9. $a = p + \frac{prn}{365}$ (p)

10. $S = vt - \frac{1}{2}gt^2$ (v)

11. By substituting in your answer to Example 2 find the height of a room in which the length is 16', the width is 14', and the surface of the four walls totals 510 square feet.

12. Find the altitude of a hollow cylinder in which $R = 4''$, $r = 3''$, and $V = 110$ cubic inches. (Use your answer to Example 7.)

13. The formula for changing from Centigrade reading to Fahrenheit reading is $F = \frac{9}{5}C + 32^\circ$ (see page 189). What is the Fahrenheit reading when the Centigrade reading is 25° ? -40° ?

14. Change the subject of the formula in Example 13 to C and express the right member in the factored form.

15. The melting point of iron is 2732° F. What is the equivalent Centigrade reading?

16. In the formula $E = \frac{C}{R + nr}$ what is the effect on E if C is increased, the values of the other letters being unchanged? if R is increased? if C is decreased? if r is decreased?

EXERCISE 85

Solve each of these equations or formulas for the letter indicated, and check:

$$1. \quad mx - 5n = 5m - nx \quad (x)$$

$$2. \quad (2x + 3h)^2 - (2x - 3k)^2 = 3(h^2 - k^2) \quad (x)$$

$$3. \quad \frac{y}{m} = b - y \quad (y)$$

$$4. \quad \frac{a}{z} + \frac{b}{z} = 1 + \frac{b}{a} \quad (z)$$

$$5. \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad (f)$$

$$6. \quad \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad (p)$$

$$7. \quad 3x - \frac{x - 3b}{a} = 9b \quad (x)$$

$$8. \quad S = vt + \frac{1}{2}at^2 \quad (a)$$

$$9. \quad h = \frac{k + i}{1 + i} \quad (i)$$

$$10. \quad C = \frac{E}{R + nr} \quad (n)$$

$$11. \quad \frac{x}{b - x} = \frac{a}{c} \quad (x)$$

$$12. \quad \frac{3mx + n^2}{x^2 - m^2} + \frac{n}{x + m} = \frac{2n}{x - m} \quad (x)$$

$$13. \quad \frac{x}{a} + \frac{x}{b} = a + b \quad (x)$$

$$14. \quad \frac{7y + 6a}{y^2 - 2ay - 15a^2} - \frac{7}{y - 5a} = \frac{5}{y + 3a} \quad (y)$$

$$15. \quad \frac{2m + 3c}{2m - 5c} + \frac{2m + 5c}{2m - 3c} = 2 \quad (m)$$

Oral Problems

1. What will represent a fraction whose numerator is n and whose denominator is d ?

2. What does this fraction become if numerator and denominator are each increased by 2?

3. Using n for the numerator, represent a fraction whose denominator is 3 more than the numerator.

4. If 5 is added to each term of the fraction in Example 3 what does the fraction become?

5. The same number n is added to each term of the fraction $\frac{9}{11}$; the new fraction is equal to $\frac{8}{9}$. Express these statements as an equation.

6. The larger of two numbers is 15 more than the smaller. If the larger number is divided by the smaller, the quotient is $1\frac{2}{3}$. Make the equation.

7. Express as an equation: D divided by d gives q for a quotient and r for a remainder.

8. What is the formula for distance in terms of rate and time?

9. At 75 miles an hour how many hours would it take the dirigible Shenandoah to go a distance of 300 miles? of d miles?

10. If a steamship goes 480 miles in 24 hours what is its average rate per hour? if it goes d miles in h hours?

11. What is the reciprocal of each of the following: $7, \frac{1}{6}, -4, x, \frac{1}{2}a$?

12. If there are x boys in a class of 35 pupils how many girls are there?

13. Separate 36 into two parts, one of the parts being n .

EXERCISE 86

PROBLEMS

1. I am thinking of a number such that the difference between its fourth and sixth parts is 15. What is the number?

2. Find that number the sum of whose third, fourth, and fifth parts is 47.

3. Divide 96 into two such parts that one-fifth of the larger part may be equal to one-third of the smaller.

4. Separate 76 into two parts so that the sum of one-fifth of the larger and one-sixth of the smaller is 14.

5. Two numbers differ by 12. One-third of the smaller exceeds one-fourth of the larger by $\frac{1}{3}$. Find the numbers.

6. 68 is four-fifths of what number?

7. What number increased by $\frac{3}{4}$ of itself gives 63?

8. The difference between a certain number and 20% of itself is 36. What is the number?

9. A farmer wishes to raise 300 chicks, but he estimates that only three-fifths of the eggs which he puts in the incubator will hatch out into chicks. How many eggs should he put in the incubator?

10. How could a father divide \$350 between his son and daughter in such a way that the son receives $\frac{3}{4}$ as much as the daughter?

11. Of what number is it true that the quotient, when the number is divided by 4, is 10 less than when the number is divided by 12?

12. Separate 125 into two parts such that if the larger is divided by the smaller, the quotient is 3 and the remainder 5.

13. The numerator of a certain fraction is 7 less than the denominator. If 4 be added to the denominator, the value of the resulting fraction is $\frac{1}{2}$. What is the fraction?

14. What number must be added to each of the terms of the fraction $\frac{5}{7}$ to make the resulting fraction equal to $\frac{5}{6}$?

15. Find that number whose reciprocal is equal to the sum of the reciprocals of 4 and 12.

16. What is the number whose reciprocal is equal to the difference between the reciprocals of 6 and 10?

17. The sum of a certain number and its reciprocal is 5 more than the number itself. What is the number?

18. Separate 127 into two parts such that the difference between one-half of the smaller part and one-fifth of the larger part is 18.

19. I am thinking of two numbers whose sum is 520. If the smaller number is divided by the larger number the quotient is $\frac{1}{7}$. What are the numbers?

20. The smaller of two numbers is 51 less than the larger. If the larger is divided by the smaller the quotient is 4. Find the numbers.

21. John can run 100 yards in 45 seconds less time than he can walk 100 yards. If he can run four times as fast as he can walk, how many yards can he run in one second?

22. The numerator of a certain fraction exceeds the denominator by 5. If 2 be subtracted from the numerator, the sum of the resulting fraction and $\frac{1}{2}$ is equal to the original fraction. Find the fraction.

23. Separate 782 into two parts, so that if the larger is divided by the smaller, the quotient is 4 and the remainder 2.

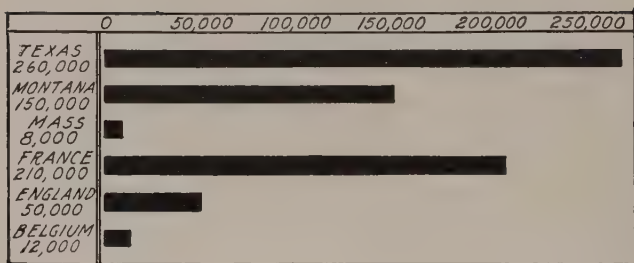
24. There are two fractions each of which has 1 for a numerator; their denominators differ by 1. The sum of the two fractions is 5 times the product of the two fractions. Find them.

25. The sum of the numerator and the denominator of a certain fraction is 26. If 1 be subtracted from each term of the fraction, the value of the resulting fraction is $\frac{1}{5}$. Find the fraction.

CHAPTER XI

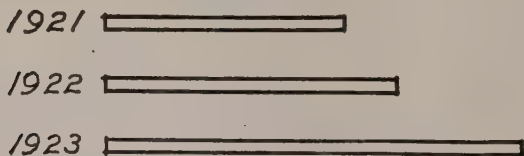
RATIO AND PROPORTION

BAR GRAPHS AND RATIO



The relative sizes of two or more different numbers of the same kind are vividly compared by means of bar graphs. The length of each bar is here scaled to represent the area of a state, each inch of its length corresponding to 100,000 square miles of area. The smallness of Massachusetts is strongly emphasized here in contrast with the huge area of Texas.

EXERCISE 87



1. The first bar is $1\frac{1}{4}$ " long and represents a merchant's sales in 1921, which were 50,000. Measure the other bars and determine what the sales were in 1922 and in 1923.

2. Draw bars to represent the population of the states given below. Use a scale of $\frac{1}{10}$ " to 100,000 persons. Contrast with the area graph above.

Massachusetts.....	3,850,000
Montana.....	550,000
Texas.....	4,650,000

3. In the entering class of a high school there were

10 pupils 12 years old
56 pupils 13 years old
95 pupils 14 years old
50 pupils 15 years old
17 pupils 16 years old
2 pupils 17 years old

Draw six vertical bars, spacing them $\frac{1}{2}$ inch apart, to represent the number of pupils of the various ages. Scale 1" to 20 pupils.

4. In 1920 there were in the United States 36,000,000 persons either foreign-born or of foreign-born parents.

<i>Country of Origin</i>	<i>Number of Persons</i>
Norway.....	1,000,000
Hungary.....	1,100,000
Sweden.....	1,500,000
England.....	2,300,000
Canada.....	2,600,000
Austria.....	3,100,000
Italy.....	3,300,000
Russia.....	3,900,000
Ireland.....	4,100,000
Germany.....	7,300,000

Choose a suitable scale and draw ten bars.

5. Draw bars to represent the combined value of farm crops and dairy products in 1920 in five states. Scale 1" to \$200,000,000.

Virginia.....	\$330,000,000
New York.....	640,000,000
California.....	700,000,000
Illinois.....	1,000,000,000
Iowa	1,020,000,000

6. Draw a bar to represent the weight per cubic foot of each of the materials named:

<i>Material</i>	<i>Weight of a Cubic Foot in Pounds</i>
White Pine.....	30
Maple.....	45
Ice.....	57
Water.....	$62\frac{1}{2}$
Aluminum.....	160
Granite.....	170

7. On a scale of 1" to \$100 draw two bars to represent a profit last week of \$300, and this week of \$75. The first bar is how many times as long as the second?

"Last week our profit was \$300; this week it is only \$75." To compare these two numbers the most instructive questions to ask are, "How much more did we gain last week than this?" and, "Last week's profit was how many times as large as this week's?" The first question asks for the difference between the two numbers, and the answer may be found by a subtraction. The second question asks for the quotient or ratio of the two numbers, and the answer is found by a division.

The *ratio* of two numbers is the quotient obtained by dividing the first number by the second. The ratio of 10' to 3' is written $10' : 3'$ and is equal to $10 \div 3$ or $\frac{10}{3}$. The

two numbers must be of the same kind; for example, one cannot speak of the ratio of five boys to two baseball games. The first term of the ratio is the dividend and corresponds to the numerator of a fraction. The second term of the ratio is the divisor and corresponds to the denominator of a fraction. The principles used in working with fractions also apply to work with ratios. The ratio sign ($:$) is often used as a division sign instead of ($\div$).

EXERCISE 88

1. How can you write the ratio of 12 to 5, using the ratio sign? Using the division sign?
2. Write the ratio of 4 to 16 in three different ways.
3. Express the fraction $\frac{3}{5}$ as a division and as a ratio.

Express each of the following ratios as a fraction, and reduce the fraction to its lowest terms:

- | | |
|--------------------------------|----------------------------|
| 4. The ratio of 8 to 6 | 10. 5 m to 10 q |
| 5. The ratio of 12" to 36" | 11. 25% to 75% |
| 6. The ratio of \$15 to \$9 | 12. 4 feet to 2 yards |
| 7. $\frac{1}{2} : \frac{1}{4}$ | 13. $3x \div x^2$ |
| 8. 1000 to 125 | 14. $\pi a^2 \div \pi b^2$ |
| 9. a^2 to ab | 15. $2\pi r_1 : 2\pi r_2$ |
16. An inch to a foot
 17. A square inch to a square foot
 18. An ounce to a pound
 19. A pound to an ounce
 20. Multiply both terms of the ratio $1\frac{1}{2}$ to 1 by 2.
 21. Multiply both terms of the ratio 0.4 to 7 by 10.
 22. Divide both terms of the ratio 5 : 2 by 2.
 23. Divide both terms of the ratio 3 : 10 by 3.
 24. Copy, and complete: "Both terms of a — may be — or — by the same number without changing the value of the —."

25. Divide the terms of each ratio by such a number that the first term shall be 1:

$$(a) \quad 4 : 100$$

$$(c) \quad 2 : 33$$

$$(b) \quad 4 : 5$$

$$(d) \quad 2\frac{1}{2} : 1$$

26. Reduce the ratio $3 : 8$ to an equivalent ratio whose first term is 1.

27. Change the following ratios so that the second term in each case shall be 1:

$$(a) \quad 30 \text{ to } 5$$

$$(b) \quad 3 \text{ to } 2$$

$$(c) \quad 22 \text{ to } 7$$

$$(d) \quad 1 : 5$$

$$(e) \quad 1728 : 231 \text{ (two significant figures)}$$

28. Write the ratio of $n + 1$ to n if (a) $n = 4$, (b) $n = 10$, (c) $n = \frac{1}{2}$.

29. Write the ratio of $(n + 1)^2$ to n^2 if (a) $n = 4$, (b) $n = 10$, (c) $n = \frac{1}{2}$.

30. Express in lowest terms the ratio to 360 days of (a) 60 days, (b) 72 days, and (c) 365 days.

31. Express each of the following ratios as an example in division, and divide:

$$(a) \quad \frac{3}{8}'' : \frac{1}{2}''$$

$$(d) \quad 6\frac{1}{4} \text{ to } 100$$

$$(b) \quad 1\frac{1}{3} : 1\frac{1}{2}$$

$$(e) \quad \frac{4}{5} : \frac{5}{4}$$

$$(c) \quad 2\frac{2}{3} : 4$$

$$(f) \quad \frac{a}{b} : \frac{b}{a}$$

32. Simplify $c^2 - h^2 : c^2 + ch$. Check when $c = 2$, $h = 1$.

33. If $a = 10$ and $b = 20$, what is the ratio of a to b ?
of a^2 to b^2 ? of $\frac{1}{a}$ to $\frac{1}{b}$?

34. Expressions such as "Two out of three" and "99 out of every 100" illustrate merely a different way of stating

ratios. Each of the ratios stated would mean how many pupils out of a school of 600?

35. In 1923 one person out of eight in New Jersey owned an automobile. How many automobiles were there in New Jersey if the population was 3,600,000?

Write each ratio as a fraction, and solve:

36. $3n : 5 = 6$

40. $x \div 12 = 3.1$

37. $2 : w = 4$

41. $s + 4 : s = 2$

38. $1 \div x = 10$

42. $5 = a : 180 - a$

39. $l : 7'6'' = \frac{2}{3}$

43. $2x + 1 : 3 = x$

EXERCISE 89

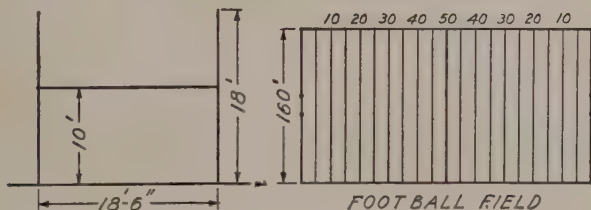
1. Express the ratio of the width to the length of

(a) a rug $9' \times 12'$

(b) a page of a book 12 centimeters $\times$ 20 centimeters

(c) a photograph $2\frac{1}{4}'' \times 3\frac{1}{4}''$

2. In each part of Example 1 express the ratio of the length to the width.



3. Express the ratio of the height of the cross-bar above the ground to the height of a post.

4. What is the ratio of the width of the football goal between the two posts to the height of the cross-bar?

5. What is the ratio of the height of a post to the height of the cross-bar?

6. The parallel lines marking off the football field are 5 yards apart. How long is the field between the goal-lines?

7. What is the ratio of the length of the field to the width? What is the ratio of the width to the length?

8. Express the ratio of the width of a goal (18'6") to the width of the field.

9. What is the perimeter of the field in feet? in yards?

10. The diagram of a square flower-bed is drawn on a scale of $\frac{1}{8}$ " to a foot. Measure the side of the diagram and so find the side of the flower-bed. What is its area? its perimeter?

SQUARE
FLOWER
BED

RECTANGULAR
FLOWER BED

11. The rectangular bed is drawn on the same scale as the square. What is its length? width? area?

12. The rectangular bed and the square bed have the same area. Do they have the same perimeter? What is the ratio of their perimeters?

13. Find the ratio of the perimeter of a rectangle $16' \times 4'$ to the perimeter of a square having the same area. Make a sketch.

14. A certain square has the same perimeter as a rectangle $9'' \times 3''$. Find the ratio of the area of the rectangle to the area of the square.

15. A rectangle and a square have the same perimeter. The side of the square is $36''$. If the length of the rectangle is twice the width, what is the ratio of the area of the square to the area of the rectangle?

A ratio is frequently best expressed as a decimal. Any ratio can be changed to a decimal by actually performing the division which is indicated.

Example: Express the ratio $7 : 22$ as a decimal (to three figures) and as a per cent.

Solution: $7 : 22$ means $7 \div 22$ or 7 divided by 22.

Divide 7 by 22:

$$\begin{array}{r}
 .318 \\
 22 \overline{) 7.000} \\
 \underline{66} \\
 40 \\
 \underline{22} \\
 180 \\
 \underline{176} \\
 4
 \end{array}$$

Check:

$$\begin{array}{r}
 .318 \\
 \underline{22} \\
 636 \\
 \underline{636} \\
 6.996 \\
 \underline{6.996} \\
 .004 \\
 \underline{.004} \\
 7.000
 \end{array}$$

The ratio 7 : 22 is therefore approximately equal to .318 or 31.8%.

EXERCISE 90

Express each of these ratios (a) as a decimal, (b) as a per cent:

1. $\frac{1}{2}$

4. $\frac{23}{20}$

2. $\frac{3}{4}$

5. Four out of five

6. 11 words out of 16

3. 1 : 8

7. 1 : 7 (to three figures)

8. 100 games out of 154 (to three figures)

9. The standing of a baseball team is determined by the ratio of the number of games won to the total number of games played. This ratio is usually expressed as a three-place decimal. If Detroit has won 4 games and lost 2, the ratio is .667. Find the ratio for Cleveland if that team has won 5 games and lost 7.

10. Find the standing of each team, and rearrange in rank:

	<i>Won</i>	<i>Lost</i>	<i>Standing</i>
St. Louis.....	30	21	.588
Brooklyn.....	35	15	
Cincinnati.....	16	32	
Philadelphia.....	27	21	
Pittsburg.....	29	23	

Express each of the following as a ratio, in the form of a common fraction in lowest terms:

11. $2\frac{1}{2}\%$

Solution: $2\frac{1}{2}\%$ means $.02\frac{1}{2}$ or $.025$. $.025 = \frac{25}{1000} = \frac{1}{40}$ Ans.

12. .8

16. 150%

13. .075

17. 3.14

14. 95%

18. $4\frac{1}{2}\%$

15. Thirty-five thousandths

19. Sixteen and five tenths

20. Express in per cent the ratio of a long ton (2240 lb.) to a short ton (2000 lb.).

21. Find to the nearest thousandth the ratio of a short ton to a long ton.

22. Express as a decimal the ratio of:

(a) an inch to a foot

(d) a cent to a dollar

(b) an ounce to a pound

(e) a centimeter to a meter

(c) a foot to a rod

(f) a millimeter to a centimeter

23. The statement that "the New York City death rate in 1921 was 11" means that the ratio of the number of deaths to the number of inhabitants was 11 : 1000. In a city of a million people there were 13,500 deaths; what was the death rate?

24. In 1920 the population of France was 39,000,000 and there were 830,000 births. What was the birth rate per 1000 inhabitants? (To two figures only.)

25. In 1916 the population of France was about the same as in 1921, but there were only 320,000 births. What was the 1916 birth rate?

26. In Boston the 1923 tax rate was \$24.70 per \$1000 of value. Express this as a rate per cent.

27. In Newark the 1923 tax rate was \$3.76 per \$100 of value. Express this

(a) per \$1000; (b) as a rate per cent; (c) as a decimal.

28. New Jersey has a one-mill tax for roads, or \$.001 for each dollar of value. This is equivalent to what rate per \$100? per \$1000?

29. From the tables in the back of this book calculate

(a) the number of cubic inches in a liquid quart

(b) the number of cubic inches in a dry quart

(c) the ratio of (a) to (b). (Nearest hundredth)

30. What is the circumference of a circle whose diameter is 10 inches? What is the formula for C in terms of D ?

31. What is the ratio of the circumference of any circle to the diameter? Give its value to four figures

32. Calculate to four figures the ratio of the diameter of a circle to the circumference. (Incomplete answer, .31——)

33. The ratio of a kilogram to a pound is $2\frac{1}{2} : 1$. Find to two figures the ratio of a pound to a kilogram.

34. There are 39.37 inches in a meter. What fraction of an inch is a centimeter?

35. From Example 34 find the ratio of an inch to a centimeter, to two figures.

36. The American flag was formerly made with its length and width in the ratio 1.9 to 1. What was the length of a flag 50" wide?

37. Recently the Fine Arts Commission decided on a ratio of 1.67 to 1 instead of the present 1.90 to 1. How long will a new flag be which is 50" wide?

38. What common fraction is approximately equal to the ratio 1.67 : 1?

Find each of the following ratios:

39. The value of a dime to that of a quarter.

40. A peck to a quart.

41. The radius of a circle to the diameter.

42. A quart to a gallon.

43. The circumference of a circle to the radius.

44. The perimeter of a square to a side.
45. The surface of a cube to one of its faces.
46. The area of a circle to the square of the radius.
47. The number of inches in the length of a certain line to the number of feet in the same length.
48. The area of a square 2" on a side to the area of a square 3" on a side.
49. The perimeter of a square 2" on a side to the perimeter of a square 3" on a side.
50. Find the ratio to two decimal places of the average rates of two airplanes if one travels 2700 miles in 26 hours and the other 1700 miles in 14 hours.

BUSINESS PROBLEMS

Percentage and Interest — Oral Exercise

1. What is 50% of each of the following: 60, 302, \$50, 30 cents, 17 inches?
2. What is 10% of each of the following: 250, \$70, 37.3, 90 cents, .7854?
3. Find 6% of 200.
4. Find 7% of 100 x .
5. If $p = br$ find the value of p if $b = \$650$ and $r = 20\%$.
6. Helen spelled correctly 70 per cent of a total of 20 words. How many words were correct?
7. George spelled correctly 15 words out of 20. What per cent was this?
8. A high school baseball team won 9 games out of 15. What per cent of its games did it win?
9. Solve: 50% of $x = 17$.
10. Solve: 25% of $x = 60$.
11. Solve: $.06x = \$12$
12. What common fraction is equal to 50%? 75%? 20%? 4%? 40%? $12\frac{1}{2}\%$? $16\frac{2}{3}\%$?

13. Express in per cent the value of each fraction: $\frac{1}{4}$, $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{3}$, $\frac{4}{5}$, $\frac{2}{3}$.

14. Two years ago Ida weighed 80 lb. Now she weighs 100 pounds. What per cent has her weight increased in two years?

15. A list of groceries which could have been bought for \$20 in 1913 costs \$32 in 1924. What per cent did the cost increase?

16. In 1908 the same list of groceries could be bought for \$16. What per cent did the cost increase from 1908 to 1913? from 1908 to 1924?

17. From the formula $p = br$ find p if

(a) $b = \$2000$ and $r = 1\%$

(b) $b = \$2000$ and $r = \frac{1}{2}\%$

(c) $b = \$2000$ and $r = 0.6\%$

18. A bank charges $\frac{1}{3}\%$ for collecting a \$3000 check from a distant city. What is the amount charged?

19. What is the interest on \$1000 for one year at 6%? at 5%? at $4\frac{1}{2}\%$

20. Using the formula $i = prt$, find i if $p = \$200$, $r = .06$, $t = 2$.

21. From the same formula find i if $p = \$50$, $r = 4\%$, and $t = \frac{1}{2}$.

22. If the interest on \$500 for one year is \$20, what is the rate of interest?

23. If $i = \$60$, $p = \$1000$, and $t = 2$, what is the value of r ?

EXERCISE 91

1. A dealer gains 8% on an automobile which cost him \$1200. Letting $b = \$1200$ and $r = 8\%$ write a formula for the gain. Find the gain. Write a formula for the gain in terms of the cost c , and the rate of gain r .

2. Write a formula for the selling price (s) in terms of cost (c) and gain (g). Find the dealer's selling price on the automobile.

3. Write a formula for the selling price in terms of c and r . Find the selling price if $c = 30$ cents and $r = 40\%$.

4. Using the formula find at what price a furniture dealer should sell a chair which cost \$15 in order to gain 30%.

5. Using the formula $s = b - br$ find at what price Mrs. Cole should sell her automobile which cost her \$900 last year but has depreciated 35%.

6. Change the subject of the formula $s = b - br$ to r .

7. Using the preceding answer find what per cent was lost by Mr. Blake who bought a share of stock for \$80 and had to sell it for \$64.

8. Using the same formula find what rate of discount was allowed a garage which purchased for \$245 a machine listed in a catalogue at \$350.

9. Change the subject of the formula $s = b + br$ to b .

10. Using the preceding answer find the cost of a house which was sold for \$7500 at a profit of 20%.

11. By the same formula find what a bicycle dealer paid for a tire which he afterward sold for \$4.50 at a profit of 50%.

12. Solve for b : $b + .15b = \$66$

13. Solve for c : $.06c = \$33$

14. Solve for x : $8\% \text{ of } x = \$4.40$

15. Solve for c : $c - .05c = \$304$

16. Solve the formula $i = prt$ for each of the letters in turn.

17. Find the interest on a \$500 Liberty Bond for one year at $3\frac{1}{2}\%$.

18. Find the semi-annual interest on a mortgage of \$5000 at 6%.

19. What principal would earn \$1500 interest in one year at 4%?

20. (a) Express in words the meaning of the formula $a = p + i$. (b) Solve the formula $a = p + prt$ for p .

21. What principal would amount to \$1680 in 2 years at 6% interest?

22. Change the subject of the formula in Example 20 to r .
23. At what rate of interest would \$1200 amount to \$1398. in 3 years?
24. Using the formula $i = pr \cdot \frac{n}{360}$ find the interest at 6% on \$400: (a) for 60 days; (b) for 6 days .
25. Using the formula $i = pr \cdot \frac{n}{365}$ find the interest on \$7300 for 40 days at $4\frac{1}{2}\%$.

Supply all the missing numbers:

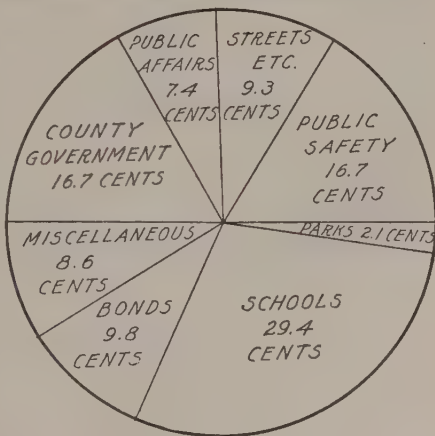
	COST	RATE OF GAIN	GAIN	SELLING PRICE
26.	24¢	100%		
27.	\$325	10%		
28.	\$8000			\$10,000
29.		40%	\$18	
30.		20%		\$900

CIRCULAR GRAPHS — MEASURING ANGLES

These sectors of this circle show how Newark spends each dollar raised by taxation.

Such graphs are useful in representing the relative size of the parts which go to make up a whole, — in this case the whole of the city's expenditures.

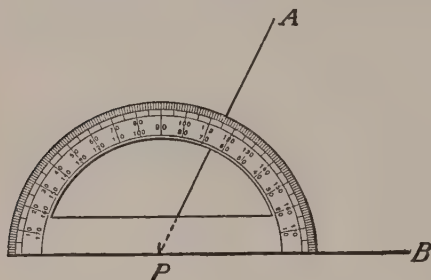
If we were to represent one-tenth of the expenditures by a sector, the sector would be $\frac{1}{10}$ of the



complete circle and the angle would be $\frac{1}{10}$ of 360 degrees or 36° .

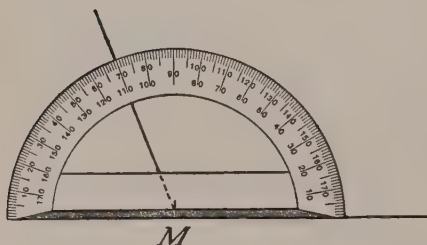
How many degrees would there be in the angle of a sector which is $\frac{1}{6}$ of the whole circle? $\frac{1}{8}$ of the circle? $\frac{2}{9}$ of the circle? 5% of the circle?

To measure the angles in degrees, so as to draw the various



sectors, each student needs to supply himself with an instrument called the protractor.

To measure the angle at *P* the protractor has been placed with its center at *P* and the straight edge of the protractor along one side of the angle, the side *PB*. The opening from one side of the angle *PB* to the other side *PA* is seen to be 65° . The second diagram illus-



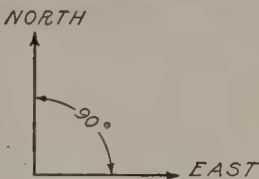
trates the use of the protractor in measuring the angle at *M*, which is found to be 110° .

If the sides of the angle to be measured are not long enough to extend past the circular edge of the protractor, the sides should be prolonged.

EXERCISE 92

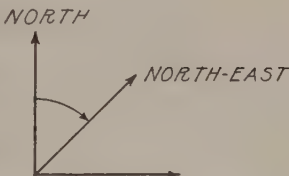
1. What fraction of the whole circle would be a sector whose angle is 90° ? 150° ? n° ?

2. If you face north and then turn to the right until you face east, you have turned through a right angle, or an angle of 90° . Suppose you turn from north to northeast, only halfway around toward the east, through how many degrees do you turn?



3. How many degrees between south and west? between south and northwest?

4. How many degrees in the angle formed by the two hands of a clock at 3 o'clock? at 1 o'clock? at 9 o'clock? at 4 o'clock?



5. Divide a circle into four sectors with the protractor so that the angles shall be 65° , 110° , 80° , and 105° .

6. Divide a circle into five equal sectors, each angle to be $\frac{1}{5}$ of 360° .

7. Mark on a circle the five points of division obtained as in Example 6, and by joining alternate points of division form a five-pointed star.

8. In a school of 600 pupils there were 250 in the first-year class, 150 in the second-year class, 110 in the third-year class, and the rest were seniors. Calculate the angles for each class ($\frac{250}{600}$ of $360^\circ = ?$) and draw a circular graph.

9. The annual budget of a family with an income of \$1500 was \$350 for rent, \$550 for food, \$300 for clothes, \$100 for recreation and vacation, \$50 for fuel and light, \$50 for insurance, and \$100 for incidentals. Draw a circular graph.

When numbers are not so simple as those in the preceding example, it is usually best to find what per cent each part is of the whole, and take that per cent of 360° for your angle.

10. What per cent of the total population of the United States, 106 million, are

- (a) the foreign-born, 14 million?
- (b) the native-born of foreign parents, 22 million?
- (c) the colored, 11 million?
- (d) the native whites, 59 million?

11. Draw a circular graph of the data in Example 10.

PROPORTION

Two equal ratios form a **proportion**, — *e.g.* $5 : 3 = 10 : 6$. The first and last terms, 5 and 6, are called the **extremes** and the other terms the **means**. A proportion is an equation, and may be transformed or solved in the same way as any other equation.

We may represent any proportion by the equation $\frac{a}{b} = \frac{c}{d}$

Multiply both sides by the l. c. d., $(bd) \frac{a}{b} = (bd) \frac{c}{d}$

$$\text{or } ad = bc$$

From this there follows a most important principle: *In any proportion the product of the extremes is equal to the product of the means.*

EXERCISE 93

1. In the proportion $5 : 3 = 10 : 6$, what is the product of the extremes? of the means?

2. In the proportion $8 : x = 4 : 3$, what products are equal? Solve for x .

Solve each of the following proportions for x :

3. $15 : x = 12 : 8$

6. $x : x + 5 = 3 : 4$

4. $a : x = b : a$

7. $\frac{2}{3} : 4 = x : 6$

5. $p + q : x = 1 : p - q$

8. $m : n = p : x$

9. Write a proportion using the numbers 3, 12, 4, and 9 in a correct order.

10. Write any proportion in which the product of the extremes shall be 24.

The principle that the product of the means is equal to the product of extremes helps to investigate the truth or falsity of a supposed proportion.

11. In each of the following find the products of the extremes and of the means, and tell which are correct proportions and which are false:

$$(a) 1 : 2 = 3 : 4$$

$$(b) 3 : 5 = 2 : 3$$

$$(c) \frac{1}{2} : \frac{1}{5} = 2 : 5$$

$$(d) \frac{.005}{1} = \frac{1}{200}$$

$$(e) 16\frac{1}{2} : 100 = 1 : 6$$

$$(f) a : b = a^2 : b^2$$

12. We can form true proportions if we take care that the product of the means shall be equal to the product of the extremes. Form a proportion using the numbers $5 \times 6 = 3 \times 10$.

13. Form as many different proportions as you can, using the numbers 5, 6, 3, and 10.

14. If $5x = 3y$, complete the proportion $x : y = ? : ?$.

15. Given that $4x = 7y$ find the ratio $x : y$.

16. If $DS = ds$, write a proportion commencing with D .

17. If $6w = w'$, what is the ratio of w to w' ?

Problems Solved by Proportion

Problems may sometimes be solved by means of proportion.

Example: Mr. Morton bought a 150-pound sack of potatoes for his home. At the end of 3 weeks he found that 35 pounds had been used. At the same rate how long would the entire sack last?

Solution: Let n = the number of weeks

$$35 \text{ pounds} : 150 \text{ pounds} = 3 : n$$

$$35n = 450$$

$$n = 13 + \text{Ans.}$$

EXERCISE 94

1. An iron bar 6' long weighs 110 lb. What would be the weight of a part of the bar, 2' long?

2. If 12 men can dig in a day a trench for a water-main 75' long, how many days would it take them to dig a trench 400' long?

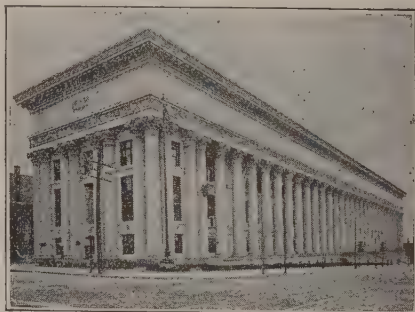
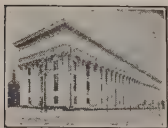
3. If a steamship burns 500 tons of coal on a voyage of 600 miles, how much coal would be needed for a voyage of 2000 miles?

4. A cylindrical water tank 15' deep holds 10,000 gallons. How many gallons are in the tank when the water is only 7' deep?

5. If 12 bananas cost 40 cents, what would 8 bananas cost?

6. If the railway fare for a journey of 250 miles is \$8.00, what would be the fare for a journey of 400 miles?

7. If a photograph $4'' \times 3''$ is enlarged by a photographer so that the length of the enlargement is $10''$, what is the width of the enlargement?



8. If the length and width of the photograph were l and w and of the enlargement L and W , what proportion would be true?

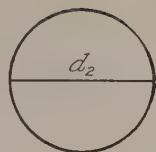
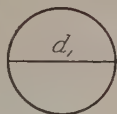
9. Show that the circumferences of two circles are to each other as their diameters:

Solution: Let d_1 and d_2 be the diameters of any two circles.

From the formula for the circumference of a circle we have

$$C_1 = \pi d_1$$

$$\text{and } C_2 = \pi d_2$$



Dividing member by member, $\frac{C_1}{C_2} = \frac{\pi d_1}{\pi d_2}$

$$\therefore \frac{C_1}{C_2} = \frac{d_1}{d_2}$$

We say that the circumferences of any two circles are in the same ratio as their diameters; or more briefly, two circumferences are proportional to their diameters.

10. Show by equations similar to those in Example 9 that any two circumferences (C_1 and C_2) are proportional to their radii (R_1 and R_2).

11. State a proportion between the weights (w_1 and w_2) of two iron bars of different lengths (l_1 and l_2), each bar being 1" square.

12. Express by a proportion: The weights (w_1 and w_2) of two discs, cut from the same piece of sheet metal, are proportional to the areas of the discs (A_1 and A_2).

13. Express by a proportion: The weights of two coils of the same kind of copper wire are proportional to the lengths of wire in the two coils.

14. Sometimes a statement having a similar meaning is made in the singular instead of the plural: The number of tons of coal burned on a voyage by the freight steamer *Hog Island* is proportional to the length of the voyage in miles. Express this fact for two voyages as a proportion, using T_1 , T_2 , m_1 , m_2 .

15. State by a similar proportion the fact that the number of gallons of gasoline (g_1 , g_2), used by an automobile is proportional to the number of miles traveled.

Ways of Expressing the Proportional Relation

The words *is proportional to* are sometimes replaced by *varies directly as* or *varies as*. Thus "Two circumferences are proportional to their diameters," "The circumference of a circle is proportional to its diameter," and "The circumference of a circle varies as its diameter" all mean the same thing.

Each is expressed by the proportion: $\frac{C_1}{C_2} = \frac{d_1}{d_2}$.

EXERCISE 95

1. Express as a proportion that the distance traveled by a train varies as the time. (Use d_1, d_2, t_1, t_2 .)

2. Express as a proportion the fact that one's fare (f_1, f_2) on a railroad varies as the distance.

3. Express as a proportion: The amount of the receipts at a theatre varies as the number of persons attending.

4. Express as a proportion: The price of flour varies with the price of wheat.

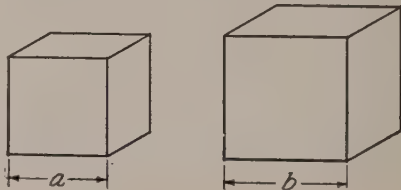
5. Translate into a formula: The Fahrenheit temperature less 32° is to the Centigrade temperature as 9 is to 5.

6. Show by a method like that of Example 9, in the previous exercise, that the areas of two circles are to each other as the squares of their radii.

7. In the proportion $A_1 : A_2 = D_1^2 : D_2^2$ find A_1 if $A_2 = 12.56$, $D_1 = 6$, and $D_2 = 4$.

8. What is the area of the total surface of a cube 2" on an edge? 3" on an edge? the ratio of their surfaces?

9. Write a proportion comparing the ratio of the total surfaces of two cubes (S_1 and S_2) with the ratio of their edges (a and b).



10. Fill in the spaces: The total surfaces of two cubes are to each other as the — of their —.

11. Using the formula $S = \pi D^2$ make a statement like that in Example 6 and a proportion like that in Example 7, connecting the ratio of the surfaces of two spheres with the ratio of their diameters.

12. If a 1" cube of iron weighs $\frac{1}{4}$ lb., what does a 2" cube weigh? a 3" cube?

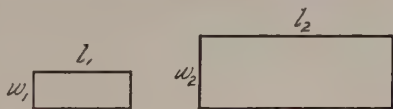
13. State as a proportion: The weights of two cubes of iron are proportional to the cubes of their edges. Write a similar proportion for the volumes of two cubes.

14. Using the formula $V = \frac{1}{6} \pi d^3$ frame a statement and a proportion concerning the ratio of the volumes of two spheres.

SIMILAR FIGURES

Two figures that have the same shape are said to be similar; their corresponding angles are equal, and their corresponding sides are proportional.

Thus the angles of the two rectangles are all of them right angles, equal in size; but the sides are



TWO SIMILAR RECTANGLES

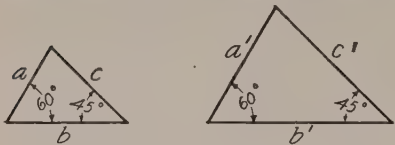
proportional, not equal. If the length l_2 is twice the length l_1 , then the width w_2 must also be twice the width w_1 .

EXERCISE 96

1. With instruments construct a rectangle $1\frac{1}{2}'' \times 2''$. Draw a second rectangle twice as long and twice as wide. Write the proportion of their lengths and widths.

2. With a protractor construct a triangle with a base 4" long, and the two angles at the ends of the base 50° and 30° . Construct another triangle with base 2" long, base angles

50° and 30°. Measure the other sides of these triangles and see if each pair of corresponding sides has the same ratio, and



TWO SIMILAR TRIANGLES

therefore that the triangles are similar.

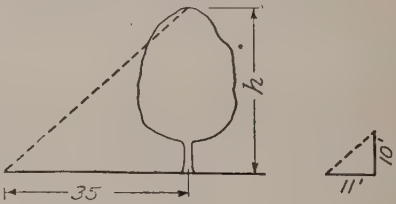
Two triangles are similar if two angles of one are equal respectively to two angles of the other.

3. With instruments construct an equilateral triangle with its sides 2" long. Construct a similar triangle having its sides $1\frac{1}{2}$ " long.

4. With instruments construct a right triangle having its arms 1" and 2" in length. Construct a similar triangle with the longer arm 3" in length; how long should the shorter arm be?

5. If a flagpole casts a shadow 32' long when a man 6' high casts a shadow $4\frac{1}{2}$ ' long, how tall is the flagpole?

6. If a 10' vertical pole casts a shadow 11' long at the same instant that a tree casts a shadow 35' long, how tall is the tree?

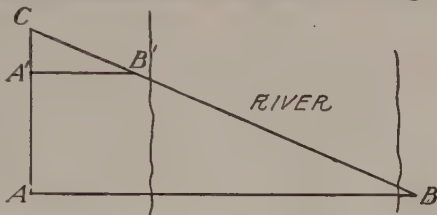


Find h , the height of a tree, in each of the Examples 7 to 11, using a diagram like that in Example 6:

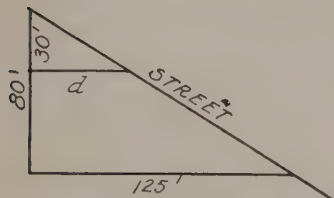
	LENGTH OF SHADOW	HEIGHT OF POLE	SHADOW OF POLE	h
7.	45'	10'	6'	
8.	78'6"	10'	18'	
9.	130.2'	10'	5.8'	
10.	55.4'	6.5'	7.1'	
11.	s	h'	s'	

12. If possible, measure the shadows of a 10' pole and of the corner of your school building and calculate the height of the latter.

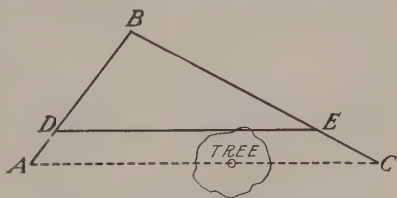
13. To measure the distance AB when B was inaccessible some boys laid off a right angle at A and a triangle $CA'B'$ similar to CAB . If CA measured 200', $CA' = 18'$, and $A'B' = 50'$ calculate by a proportion the length of AB .



14. Mr. Carr owned a triangular corner lot, 80' front and on one side 125' deep. To widen the street, the city bought from him a triangular piece, 30' front. By similar triangles find the side d of his present lot and the area of the lot.



15. The distance from A to C was to be measured where a large tree stood directly between. A stake was placed off to one side at B and the triangle BDE laid out similar to ABC . If $BD = 15'$, $BA = 20'$, and $DE = 30'$ find AC by means of a proportion.

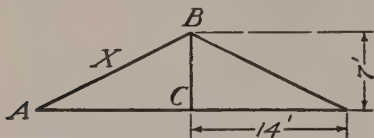


16. Draw and letter a diagram like that of Example 15. If the base $AC = 10''$, $BC = 12''$, and $BE = 9''$, find DE .

17. In the same diagram find BD if $AC = 12''$, $AD = 6''$, and $DE = 8''$. (Let $BD = x$, then $BA = ?$)

18. If the sides of a triangle are 2'', 3'', and 4'', and the smallest side of a similar triangle is 6'', find the other two sides.

19. By drawing to scale a small triangle similar to a large one, we can by means of proportion obtain unknown lengths.



Referring to the diagram of a roof, the question, "How long shall we saw the rafter AB ?" may be answered by drawing the

triangle BCA to scale $\frac{1}{4}"$ to the foot. Construct a right angle and lay off on the sides of the right angle CB and CA to scale. Measure the hypotenuse of the small triangle, and calculate the length of the rafter x .

20. On an architect's plan of the front of a building $24'$ wide and $90'$ high, the width is to be represented by a line $18''$ long. How long a line will represent the height?

21. If in a drawing of a door $36'' \times 75''$ the width of the door is represented by $2''$ what represents the height of the door?

22. A scale of $\frac{1}{4}"$ to the foot represents a ratio of 1 to ?

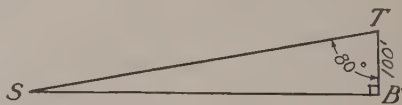
23. A scale of $\frac{3}{4}"$ to the foot represents what ratio?

24. What ratio is represented by the scale of a map $1''$ to $2000'$?

25. On a military map scale $3''$ to a mile, two villages are $7\frac{1}{2}''$ apart. How many miles is it from one village to the other?

26. A surveyor has measured the sides of a triangular lot, $200'$, $150'$, and $100'$, and wishes to find the angles. Using a scale of $1''$ to $40'$, construct a similar triangle, using compasses. With a protractor measure the three angles of your triangle.

27. A masthead observer at T , $100'$ above the water, observes a submarine at S , and



measures the angle at T , 80° . Draw the triangle BTS to scale and find how far away the submarine is. (Find BS .)

How to Find Two Numbers Which Have a Given Ratio

Two numbers which are to each other as $a : b$ may best be represented by ax and bx .

Example: Divide 96 into two parts which are to each other as 5 : 11 :

Let $5x$ = the smaller number

then $11x$ = the larger number

$$5x + 11x = 96$$

$$16x = 96$$

$$x = 6$$

$$5x = 30, \text{ the smaller number}$$

$$11x = 66, \text{ the larger number}$$

Check:

$$\frac{30}{66} = \frac{5}{11}$$

and

$$30 + 66 = 96$$

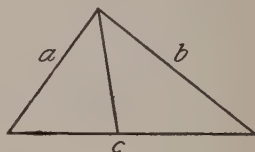
EXERCISE 97

1. Divide 200 in the ratio of 3 : 5.
2. Divide $36''$ in two parts proportional to 7 : 2.
3. The sum of two numbers is 130 and the ratio of the numbers is 3 : 7. What are the numbers?
4. The freight rate between two cities 200 miles apart is 40 cents per 100 lb. How should the 40 cents be divided between two railroads, one of which carries the freight 125 miles, the other one 75 miles?
5. Seven months after Mr. Jones had paid \$21 in advance for a year's fire insurance on his house he sold the house to Mr. Adams. How much toward the cost of this fire insurance should Mr. Adams pay over to Mr. Jones?
6. Brass is an alloy made from two parts of copper and one part of zinc. How many ounces of copper are in 16 ounces of brass? how many of zinc?
7. Air is 21 parts by volume of oxygen and 79 parts by volume of nitrogen. How many cubic feet of oxygen are there in a room containing 1500 cubic feet of air?

8. A store with 1000 square feet of floor space is rented for \$155 a month. In studying the profitableness of various departments how much of the rent should be paid by the soda fountain using 350 sq. feet? the tobacco counter using 300 sq. feet? newspapers using 300 sq. feet? candy using 150 sq. feet?

9. A block 200' long has been newly paved for \$450, the expense to be borne by abutting property owners. Divide the expense between A who owns 125 feet and B who owns 75 feet.

10. The line divides c into two parts proportional to a and b . Find the parts, (a) if $a = 9''$, $b = 12''$, $c = 14''$. (b) in terms of a , b , and c .



11. The capital stock of a business is \$16,500. How should the year's profit of \$3500 be divided between A who owns \$4500 of the stock, B who owns \$6000, and C who owns the rest?

12. In preparing a 1 : 3 : 5 (1 part cement, 3 parts sand, 5 parts gravel) mixture, how many pounds of cement, sand, and gravel would be used for a ton of the mixture?

13. The perimeter of a triangle is 48" and the three sides are as 3 : 6 : 7. Find the sides.

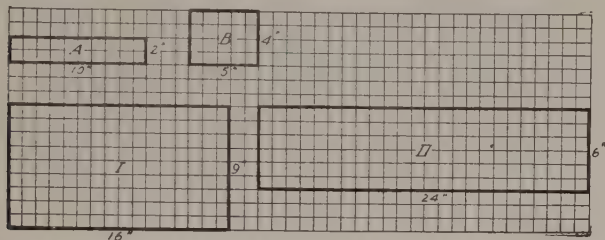
14. The perimeter of a triangle is d'' and the three sides are to each other as $a : b : c$. Find the sides.

QUANTITIES WHICH ARE INVERSELY PROPORTIONAL

The area of rectangle A is the same as that of rectangle B .

Which rectangle has the greater length? the greater width? The rectangle that has the greater length has the smaller width. Rectangle A is twice as long as rectangle B , but only one-half as wide.

Show that the area of either I or II is one square foot. What is the ratio of the length of I to the length of II? What is the ratio of the width of I to the width of II? Rectangle I is only two-thirds as long as rectangle II, but its width is



three-halves of the width of II. Since one of these ratios is the inverse of the other, the lengths and widths of any two rectangles having the same area are said to be *inversely proportional*.

EXERCISE 98

1. The products 16×9 and 6×24 are equal. Write a proportion, using the four numbers, 16, 9, 6, and 24.

2. If we have two rectangles of the same area, then $l_1 w_1 = l_2 w_2$. Fill in the missing terms of the proportion $l_1 : l_2 = \text{---} : \text{---}$.

3. Write a proportion using the lengths and widths of the rectangles A and B above.

4. A rectangle 4" wide has the same area as a rectangle $18" \times 8"$. Find the length of the first rectangle by a proportion.

5. If you had \$3 to spend for candy, how many pounds could you buy at 25¢ a pound? How many pounds at 30¢ a pound? At 60¢ a pound?

6. Copy, using the words "greater" or "smaller" to fill each blank space: The — the price per pound the — the number of pounds that can be bought for the same money.

7. In Example 5, suppose that at the price of p_1 cents a pound you could buy n_1 pounds, and that at p_2 cents you could buy n_2 pounds. Write a proportion showing that n_1 and n_2 are inversely proportional to the prices p_1 and p_2 .

8. A train at 50 miles an hour would travel 200 miles in how many hours? An automobile at 20 miles an hour would need how many hours to go the same distance?

9. Is the number of hours it takes to go a certain distance proportional to the rate of speed, or is it inversely proportional? Can you form a proportion based on Example 8, but using h_1 , h_2 , r_1 , and r_2 ?

10. If 10 men can dig a certain length of ditch in 6 days, how many men would have to be employed to dig that length in 3 days?

11. If 8 strips of carpet 27" wide are sufficient for the width of a certain room, how many strips of carpet 36" wide would have been sufficient?

12. If \$2 will buy 8 pounds of a certain kind of candy, how many pounds will \$6 buy? ($\$2 : \$6 = 8 : ?$) These quantities are directly proportional.

13. If d_1 dollars will buy n_1 pounds of candy and d_2 dollars will buy n_2 pounds of the same kind, what proportion is true?

14. The faster the rate at which a train travels the greater the distance it will cover in a given time. Express this in the form of a proportion (r_1 , r_2 , etc.).

15. The larger the diameter of a circle, the greater the circumference. Complete a proportion: $C_1 : C_2 = ?$.

16. The larger the diameter of a wheel the — (larger or smaller?) the number of revolutions it makes in going a mile. An automobile wheel 30" in diameter makes 1200 revolutions in going a certain distance. How many revolutions would a 36" wheel make in the same distance?

17. The larger the principal invested at a given rate of interest, the greater the income. Write as a proportion.

18. The greater the interest rate, the larger the annual income from a principal of \$1000. Express this fact as a proportion.

19. What is the relation between the time necessary for \$1000 to earn a given number of dollars interest and the rate? What proportion is true (t_1 , r_1 , etc.)?

20. A shingle laid with 4" exposed to the weather covers on the average 4×4 or 16 square inches; one laid with 5" exposed covers 4×5 or 20 square inches. To lay a certain roof with 4" of each shingle exposed required 3000 shingles. How many would have been required if 5" had been exposed?

21. If on a teeter or see-saw a boy weighing 160 pounds sits 5 feet from the point of support (the fulcrum), how far away must an 80-pound boy be seated to balance him?

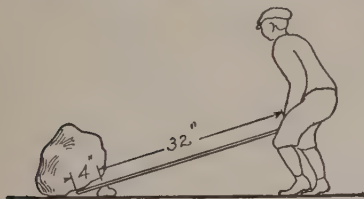


22. Complete a proportion, based on the diagram: $W : P = \quad : \quad$.

23. If Ellen weighs 96 pounds and sits 6 feet from the fulcrum, how far from it must her 120-pound brother Bob sit, to balance her?

24. Two boys weighing 125 and 100 pounds wish to balance, sitting on the ends of a board 18' long. How far shall the heavier boy sit from the fulcrum?

25. How heavily must I push down on the end of a 36" crow-bar in order to move a 400-lb. rock at the tip of the bar if the fulcrum is 4" from the tip?



10" screw-driver. What force is lifting the tack which is at the other end of the screw-driver and 1" from the point of support? Make a sketch like that for Example 25.

26. In forcing out a tack my hand pushes down with a 10-lb. push at one end of a

27. Which of the quantities named are directly proportional and which are inversely proportional:

(a) the number of pounds of potatoes you buy at 5 cents a pound and the amount you spend to buy them.

(b) the number of men to do a certain piece of work and the time necessary to finish it.

(c) the length of a rectangle of fixed width and the area of the rectangle.

(d) the length of a rectangle of fixed area and the width of the rectangle.

(e) the number of steps you take in walking a mile and the length of your step.

(f) the time necessary for an automobile to go a certain distance, and the speed of the automobile.

(g) the number of yards of ribbon you can buy for a dollar and the price of the ribbon per yard.

(h) in radio operation, the wave-length and the frequency in kilo-cycles.

CHAPTER XII

SQUARE ROOT AND RADICALS

The square root or the cube root of a monomial can often be obtained by inspection, with the aid of the principles expressed by two formulas:

(1) The square root of a product: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

(2) The square root of a quotient or fraction: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

EXERCISE 99

1. Copy these statements, based on the two formulas, and complete them:

(a) The square root of a — is equal to the — of the — of its factors.

(b) The — of a fraction is equal to the — of the — divided by the — of the —.

2. Can you write formulas for the cube root of a product and of a quotient? (Hint: $\sqrt[3]{ab} = \text{etc.}$)

Find by inspection:

3. $\sqrt{100}$

11. $\sqrt{10,000}$

4. $\sqrt{9s^2}$

12. $\sqrt{1,000,000}$

5. $\sqrt{m^2x^2}$

13. $\sqrt{\frac{x^2}{y^2}}$

6. $\sqrt{16x^4}$

14. $\sqrt{\frac{4}{9}}$

7. $\sqrt{144c^2}$

15. $\sqrt{.09R^2}$

8. $\sqrt{a^4b^6}$

9. $\sqrt{1.44}$

16. $\sqrt{\frac{a^2}{b^2c^2}}$

10. $\sqrt{.25}$

17. $\sqrt{\frac{1}{16}} y^2$

18. $\sqrt{2\frac{1}{4}}$

19. $\sqrt{11\frac{1}{9}}$

20. $\sqrt[3]{27}$

21. $\sqrt[3]{8 n^3}$

22. $\sqrt[3]{-125}$

23. $\sqrt[3]{-1000}$

24. $\sqrt{(-a)^6}$

25. $\sqrt[3]{64 a^6}$

26. $\sqrt[3]{-\frac{1}{8}}$

27. $3\sqrt{36 s^2}$

28. $4c\sqrt{4c^2}$

29. $\frac{2}{3}\sqrt{\frac{9}{16}}$

30. $\frac{a}{b}\sqrt{b^2}$

31. $\pi\sqrt{\frac{1}{4}D^2}$

32. $\sqrt{(a+b)^2}$

33. $\sqrt{3^2 + 4^2}$

34. $\sqrt{3^2} + \sqrt{4^2}$

35. Is $\sqrt{a^2 + b^2}$ equal to $\sqrt{a^2} + \sqrt{b^2}$? Test when $a = 5$ and $b = 12$.

36. $\sqrt{5^2 - 3^2}$

If $m = 36$ and $n = 64$ find the value of:

37. $\sqrt{m^2}$

39. $\sqrt{m+n}$

38. $\sqrt{m} + \sqrt{n}$

40. $\sqrt{\frac{n-m}{7}}$

SQUARE ROOT BY FACTORING

The square root of many algebraic trinomials can be obtained by factoring.

Example: Find the square root of $9c^2 - 6c + 1$.

Solution:

Factoring $9c^2 - 6c + 1$ by the cross-products method,

$$\begin{aligned} 9c^2 - 6c + 1 &= (3c - 1)(3c - 1) \\ &= (3c - 1)^2 \end{aligned}$$

$$\text{Therefore } \sqrt{9c^2 - 6c + 1} = 3c - 1 \quad \text{Ans.}$$

If a trinomial is a square,

(1) the first and third terms must each be squares; and

(2) they must be the squares of such numbers that twice the product of the numbers will be the middle term of the trinomial.

EXERCISE 100

Find the square root of each trinomial by factoring:

1. $a^2 + 10a + 25$

6. $c^2d^2 - 6cdx + 9x^2$

2. $16 + 8m + m^2$

7. $m^4 + 2m^2n^2 + n^4$

3. $4r^2 - 4r + 1$

8. $x^2 + p^2 + 2px$

4. $9 - 12N + 4N^2$

9. $a^2y^2 - 14a^2y + 49a^2$

5. $36x^2 - 60xy + 25y^2$

10. $8q^2r + 16r^2 + q^4$

Factor if possible, and state the square root of each expression if it has any:

11. $a^2 + b^2$

15. $a^2 - x^2$

12. $x^2 + xy + y^2$

16. $t^2 - 20t + 100$

13. $25y^2 - 20yz + 4z^2$

17. $4b^2 + 2b + 1$

14. $9c^2 + 6c - 1$

18. $x^3 + 2x^2y + xy^2$

Fill in the missing term so as to complete a trinomial square:

19. $n^2 + 4n + (?)$

23. $p^2 + 2pq + (?)$

20. $x^2 - 2x + (?)$

24. $x^4 - 14x^2 + (?)$

21. $y^2 + 12y + (?)$

25. $4x^2 + (?) + 25$

22. $x^2 - 18x + (?)$

26. $144 + (?) + a^2$

Factoring may also be used, in many cases, to find the square root or cube root of an arithmetical number. Oftentimes the conditions of a problem point clearly to this method as a possibility.

Examples: (a) $\sqrt{16 \cdot 49 \cdot x^2} = 4 \cdot 7 \cdot x = 28x$

(b) $\sqrt{1225} = \sqrt{25 \cdot 49} = 5 \cdot 7 = 35$

With more difficult numbers get the factors by successive short divisions.

$$\begin{array}{r}
 (c) \sqrt[3]{9261} \\
 3 \overline{)9261} \\
 \underline{3 087} \\
 3 \overline{)1029} \\
 \underline{7 343} \\
 7 \overline{)49} \\
 \underline{7}
 \end{array}$$

$$\begin{aligned}
 \sqrt[3]{9261} &= \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 7 \cdot 7 \cdot 7} \\
 &= \sqrt[3]{3^3 \cdot 7^3} \\
 &= 3 \cdot 7 \\
 &= 21 \quad \text{Ans.}
 \end{aligned}$$

EXERCISE 101

Factor and obtain the indicated root. Check by multiplication.

1. $\sqrt{4 \cdot 9 \cdot 49}$
2. $\sqrt{144 \cdot 81}$
3. $\sqrt{784}$
4. $\sqrt{1764}$
5. $\sqrt{11025}$
6. $\sqrt[3]{27 \cdot 1000 \cdot 8}$
7. $\sqrt[3]{5832}$
8. $\sqrt[3]{21952}$
9. $\sqrt[3]{27^2}$
10. $\sqrt{24^2 + 7^2}$
11. Find $\sqrt{pq}$ if $p = \frac{4}{5}$ and $q = \frac{9}{5}$
12. Find $\sqrt{mn}$ if $m = \frac{81}{17}$ and $n = \frac{25}{17}$
13. Find $\sqrt{a^2 + b^2}$ if $a = 9$ and $b = 12$
14. Find $\sqrt{R^2 - \left(\frac{s}{2}\right)^2}$ if $R = 29$ and $s = 40$

A PROCESS FOR THE SQUARE ROOT OF ANY NUMBER

In obtaining square roots why cannot the factoring method of the preceding exercise be always used?

Any number of two digits, such as 28 or 94, consists of a certain number of tens (t) plus a certain number of units (u). The square of such a number, or $(t + u)^2$, must include all the terms of $t^2 + 2tu + u^2$, which are, respectively, the

square of the tens, twice the — of the — and the —, and the — of the —.

To find the square root of 1849, consider 1849 to be $t^2 + 2tu + u^2$. Since 1849 is between 1600 and 2500 it is the square of some number between 40 and 50, or of 40 plus some units. $\therefore t = 40$ and $t^2 = 1600$.

Square root,	t	1849	<u>40 +</u>
<i>first stage:</i>	$t^2 =$	1600	
		249	$= 2tu + u^2$

Subtracting t^2 or 1600 from 1849 we have left 249, which must represent $2tu + u^2$.

Since u is small compared to t , let us consider that 249 is just a little more than $2tu$; dividing it by $2t$ will give us u and a little something over. We know that $2t$ is twice 40 or 80. Dividing 249 by 80 we get 3 and a small remainder. Set down 3 as the value of u :

Square root,	t	u
<i>second stage:</i>	1849	<u>40 + 3</u>
	$t^2 =$	1600
	$2t =$	80
		249
		$= 2tu + u^2$

Since $249 = 2tu + u^2$ it must be $u(2t + u)$. If u is 3, then $2t + u$ is equal to $80 + 3$ or 83. Then $u(2t + u)$ is $3(83)$. Multiplying 3 times 83 we get 249.

Square root,	t	u	
<i>final stage:</i>	1849	<u>40 + 3</u>	$= 43$
	$t^2 =$	1600	43
	$2t =$	80	$249 = 2tu + u^2$
	$2t + u =$	83	$249 = u(2t + u)$
			<u>129</u>
			172
			1849

In actual practice we economize by setting off the hundreds by themselves as in the example below, and by omitting unnecessary ciphers.

Example: Find the square root of 1764.

<i>Solution:</i>	<i>Check:</i>
Square root,	$\widehat{17} \widehat{64} \underline{42}$ 42
practical form:	16 42
(2 <i>t</i>) Trial Divisor = 80	84
(2 <i>t</i> + <i>u</i>) Complete Divisor = 82	168
	<u>1764</u>

In one respect this is a risky economy, for in writing the trial divisor we now have to remind ourselves that the 4 really stands for 40, and the trial divisor is 80, not 8. *To get the trial divisor, therefore, multiply the part of the root already found by 2 and annex a cipher.*

Example: Find the square root of 7225.

<i>Solution:</i>	<i>Check:</i>
	$\widehat{72} \widehat{25} \underline{85}$ 85
	64 85
Trial Divisor = 160	425
Complete Divisor = 165	680
	<u>7225</u>

To find the square root of 7225 set off the two right-hand figures. The largest square in 72 is 64, so that $t = 8$. Since the 8 really represents 80, the trial divisor is 160; etc.

$$100^2 = 10000$$

$$1000^2 = 1,000,000$$

The square of any 3-digit number must be at least as large as the square of 100 and therefore must have at least 5 digits. The square of the smallest 4-digit number, which is 1000, has 7 digits. How many digits are there in the square root of 636363? of 1234567?

If we set off periods of two figures each, starting from

the decimal point, then for each of the periods there will be one figure in the answer. Thus we would write $\widehat{1} \widehat{35} \widehat{79}$, $\widehat{2} \widehat{00.14}$, $\widehat{.00} \widehat{07} \widehat{29}$. To such a number as 127.6 or .037 an extra cipher is annexed, thus: $\widehat{1} \widehat{27.60}$, $\widehat{.03} \widehat{70}$

Example: Find the square root of 401,956.

Solution:

Check:

	$\widehat{40} \widehat{19} \widehat{56} \underline{634}$	634
	36	634
Trial Divisor = 120	$\overline{) 419}$	2536
Complete Divisor = 123	369	1902
Trial Divisor = 1260	$\overline{) 5056}$	3804
Complete Divisor = 1264	5056	401956

After setting off periods do not again consider the decimal point until the work is complete; at that time for each period that was set off to the right of the decimal point, point off one place in the answer.

EXERCISE 102

Find the square root of each number, and check by multiplication:

- | | |
|---------|-------------|
| 1. 1681 | 6. 15129 |
| 2. 2809 | 7. 400689 |
| 3. 841 | 8. 767376 |
| 4. 9025 | 9. 9.7969 |
| 5. 6241 | 10. 2580.64 |

Check Examples 11 to 14 by dividing the number given here by your answer:

- | | |
|-----------|-------------|
| 11. 9409 | 13. .004096 |
| 12. 14641 | 14. 2580.64 |

15. Multiply 3.14 by itself, and find the square root of the product.

16. Write any number having three digits, multiply it by itself, and find the square root of the product.

17. Do the same with a number of four digits.

18. Prove, by extracting the square root of 1310 to five significant figures, that $\sqrt{1310} = 36.194$.

Find the square root of each of the following numbers correct to three significant figures. Check by multiplication, adding to the product the last remainder:

19. $\widehat{2.00} \widehat{00}$ Ans. 1.41

20. 1000

21. 7.5

22. 75

23. 60

24. 600

25. .72. Check by division

26. .072. Check by division

27. $\frac{7}{25}$. First change to a decimal

28. $\frac{5}{6}$. To four figures

29. $3\frac{1}{3}$. To five figures

30. .0039. To the nearest thousandth.

31. What number multiplied by itself gives 408321?

32. There are two equal numbers whose product is 13924. What are they?

33. The area of a square field is 102400 square rods. How long is a side of the field? its perimeter?

34. If $\sqrt{2} = 1.4142$ find the value to three significant figures of $\sqrt{65} + \sqrt{2}$.

35. Find the value of $\sqrt{65} + \sqrt{2}$.

36. Find $\sqrt{5}$ to three figures.

37. Find $\sqrt{500}$ to three figures. What is the ratio of $\sqrt{500}$ to $\sqrt{5}$?

SIMPLIFYING RADICALS

Since $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ it is possible to simplify many radicals where the number under the radical sign has a factor that is a square.

Example 1: Simplify $\sqrt{700}$.

$$\text{Solution: } \sqrt{700} = \sqrt{100 \times 7} = \sqrt{100} \times \sqrt{7} = 10\sqrt{7}$$

By simplifying in this way, we are enabled to save ourselves much time and trouble in square root examples. In a very large number of the examples we can quickly simplify and then obtain the square root at once if we know the square root of 2 and of 3.

Example 2: Simplify $\sqrt{50}$ and find its value if $\sqrt{2} = 1.4142$.

$$\begin{aligned} \text{Solution: } \sqrt{50} &= \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2} \\ 5\sqrt{2} &= 5(1.4142) = 7.071 \quad \text{Ans.} \end{aligned}$$

Example 3: Find $\sqrt{\frac{2}{49}}$ to five figures:

$$\begin{aligned} \text{Solution: } \sqrt{\frac{2}{49}} &= \sqrt{\frac{1}{49} \cdot 2} = \sqrt{\frac{1}{49}} \cdot \sqrt{2} = \frac{1}{7} \sqrt{2} \\ \frac{1}{7} \sqrt{2} &= \frac{1}{7} (1.4142) = .20203 \quad \text{Ans.} \end{aligned}$$

The above examples suggest a definition: A radical indicating square root is said to be in its simplest form when the number under the radical sign is a whole number and has no factor which is a square.

EXERCISE 103

1. Find the square root of 300 by the usual square root process, to 5 figures.

2. Simplify $\sqrt{300}$ as in model Example 1, and find the value of the result if $\sqrt{3} = 1.7321$. Is this easier than Example 1?

3. Find $\sqrt{75}$ by the square root process, to the nearest ten-thousandth.

4. Simplify $\sqrt{75}$ and find the value of the result as in Example 2. Which took you a longer time, Example 3 or 4?

Before trying any more examples of this kind you need further practice in simplifying. Rule four columns on your paper and simplify Examples 6 to 27:

		$\sqrt{4 \cdot 5}$	$\sqrt{4} \cdot \sqrt{5}$	$2\sqrt{5}$
5.	$\sqrt{20}$			
6.	$\sqrt{12}$			
7.	$\sqrt{27}$			
8.	$\sqrt{44}$			
9.	$\sqrt{90}$			
10.	$\sqrt{1000}$			
11.	$\sqrt{48}$			
12.	$\sqrt{9a}$			
13.	$\sqrt{a^2b}$			
14.	$\sqrt{4c^2x}$			
15.	$\sqrt{\pi R^2}$			
16.	$\sqrt{D^3}$			
17.	$\sqrt{l^3w^2}$			
18.	$\sqrt{28m^4}$			
19.	$\sqrt{n^5}$			
20.	$5\sqrt{700}$			
21.	$2\sqrt{45}$			
22.	$3\sqrt{40}$			
23.	$\frac{1}{5}\sqrt{75}$			
24.	$\sqrt{\frac{4s}{9}}$			
25.	$\frac{1}{2}\sqrt{\pi D^2}$			
26.	$\sqrt{\frac{9c^4h}{16}}$			
27.	$\sqrt{\frac{3}{16}}$			

28. Given $\sqrt{2}$ as in Example 3, find in the easiest way the value of $\sqrt{200}$ and of $\sqrt{288}$ (to 5 figures).

29. Similarly find the value of $\sqrt{32}$, of $\sqrt{8}$, and of $\sqrt{18}$. Check your answers roughly by the table of square roots on page 377.

30. Simplify and find the value of $\sqrt{27}$, of $\sqrt{12}$, and of $\sqrt{48}$. (See Example 2.) Check by the tables.

The cube root of a number may be simplified if the number has a factor which is a perfect cube. Give a definition of simplest form for a radical indicating cube root.

Express in simplest form:

31. $\sqrt[3]{24}$

Solution: $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8} \cdot \sqrt[3]{3} = 2\sqrt[3]{3}$ Ans.

32. $\sqrt[3]{40}$

33. $\sqrt[3]{270}$

34. $\sqrt[3]{54}$

35. $\sqrt[3]{2000}$

36. $\sqrt[3]{a^3b}$

37. $\sqrt[3]{x^5}$

38. $\sqrt[3]{a^4b^3}$

39. $\sqrt[3]{\pi D^3}$

40. $\sqrt[3]{250m}$

41. $\frac{1}{2}\sqrt[3]{16}$

42. $\frac{1}{3}\sqrt[3]{36\pi R^3}$

43. Find $\sqrt[3]{10,000}$, given $\sqrt[3]{10} = 2.154$

The square root of a fraction frequently occasions much trouble to pupils when they use, without much thought, the second principle at the beginning of this chapter. Thus if

we attempt to find the square root of $\frac{5}{7}$ by writing $\sqrt{\frac{5}{7}} = \frac{\sqrt{5}}{\sqrt{7}}$ making use of principle 2 immediately, we are obliged to find the square root of 5 and then of 7. This gives us $\frac{2.236}{2.645}$, a result which is of little value unless we change it to a decimal by the laborious process of dividing by 2.645.

One way by which these difficulties can be avoided is to reduce $\frac{5}{7}$ to a decimal in the beginning, and then take the square root of the decimal.

Another way is to express the radical in simplest form by changing the fraction to an equivalent fraction having a perfect square as its denominator.

Example: Express in simplest form $\sqrt{\frac{5}{7}}$.

Solution: $\sqrt{\frac{5}{7}} = \sqrt{\frac{35}{49}} = \sqrt{\frac{1}{49} \cdot 35} = \frac{1}{7} \sqrt{35}$

To find the value of this result we would have to find only one square root ($\sqrt{35}$) instead of two ($\sqrt{5}$ and $\sqrt{7}$) and to divide by 7 instead of by 2.645.

EXERCISE 104

Express in simplest form:

1. $\sqrt{\frac{37}{49}}$

5. $\sqrt{\frac{22}{7}}$

2. $\sqrt{\frac{5}{9}}$

6. $\sqrt{\frac{1}{2}}$

3. $\sqrt{\frac{2}{3}}$ Ans. $\frac{1}{3} \sqrt{6}$

7. $\sqrt{\frac{3}{2}}$

4. $\sqrt{\frac{1}{7}}$ Ans. $\frac{1}{7} \sqrt{7}$

8. $\sqrt{\frac{3}{8}}$

9. $\sqrt{\frac{3a^2}{4}}$

13. $a\sqrt{\frac{1}{a}}$

10. $\sqrt{\frac{D^2}{2}}$ Ans. $\frac{D}{2}\sqrt{2}$

14. $\sqrt{\frac{A}{4\pi}}$

11. $\sqrt{\frac{1}{3m^2}}$

15. $\sqrt{\frac{4h^2}{3}}$

12. $\sqrt{\frac{x}{y^2z}}$

16. $\frac{1}{2a}\sqrt{\frac{4a}{b}}$

17. Find the value of $\pi\sqrt{\frac{l}{g}}$ if $\pi = 3.14$, $l = 25$, $g = 32$ by substituting these values and simplifying the resulting radical.

18. Simplify $\sqrt{\frac{2s}{g}}$.

19. In the formula $t = \sqrt{\frac{2s}{g}}$ find the value of t after simplification if $s = 288$ and $g = 32$.

The cube root of a fraction may be simplified by changing the fraction to an equivalent one that has a cube for its denominator:

20. $\sqrt[3]{\frac{1}{4}} = \sqrt[3]{\frac{2}{8}} = \text{etc.}$

24. $\sqrt[3]{\frac{4}{s^2}} = \sqrt[3]{\frac{4s}{s^3}} = \text{etc.}$

21. $\sqrt[3]{\frac{3}{4}}$

25. $\sqrt[3]{\frac{a^2}{b}}$

22. $\sqrt[3]{\frac{1}{9}}$

26. $m\sqrt[3]{\frac{x}{8m}}$

23. $\sqrt[3]{\frac{2}{3}}$

27. $2\sqrt[3]{\frac{\pi D^3}{6}}$

*28. In the formula $d = \sqrt[3]{\frac{64H}{n}}$ find the value of d after simplification, if $H = 120$, $n = 180$, and $\sqrt[3]{18} = 2.682$.

Simplify the radicals in Examples 29–38 and with help of the tables on page — find the value of each in the easiest way.

29. $\sqrt{200}$

34. $\sqrt[3]{80}$

30. $\sqrt{75}$

35. $\sqrt[3]{250}$

31. $\sqrt{125}$

36. $\sqrt[3]{5000}$

32. $\sqrt{\frac{1}{2}}$

37. $\sqrt[3]{\frac{1}{2}}$

33. $\sqrt{\frac{1}{5}}$

38. $\sqrt[3]{\frac{2}{25}}$

To add or subtract radicals it is necessary to change each radical to simplest form, and then add the coefficients of radicals that are alike.

Simplify and add:

39. $\sqrt{12} + 5\sqrt{3}$

Ans. $7\sqrt{3}$

40. $\sqrt{\frac{1}{2}} + \sqrt{8} - \sqrt{\frac{2}{9}}$

Ans. $\frac{13}{6}\sqrt{2}$

41. $\sqrt{\frac{1}{5}} + \sqrt{5} + \sqrt{20}$

42. $\sqrt{x} + \sqrt{4x} + \sqrt{9x}$

43. $\sqrt[3]{16} + \sqrt[3]{54} - \sqrt[3]{2}$

44. Write the sum of $\sqrt{18} + \sqrt{12} - \sqrt{27} + \sqrt{32}$ as a binomial.

MULTIPLICATION OF RADICALS.

Besides the fundamental principle that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ one who is multiplying radicals needs to bear in mind the definition of square root, which may be written as a formula:

$$\sqrt{n} \cdot \sqrt{n} = n$$

The radicals in any answer should be left in simplest form.

Example 1: Multiply $\sqrt{3}$ by $\sqrt{12}$

Solution: $\sqrt{3} \cdot \sqrt{12} = \sqrt{3 \cdot 12} = \sqrt{36} = 6$ Ans.

Example 2: Multiply $2\sqrt{5}$ by $5\sqrt{15}$

Solution: $2\sqrt{5} \cdot 5\sqrt{15} = 10\sqrt{5 \cdot 15} = 10\sqrt{5 \cdot 5 \cdot 3}$
 $= 10\sqrt{5^2 \cdot 3} = 10 \cdot 5\sqrt{3}$
 $= 50\sqrt{3}$ Ans.

Example 3: Find the product of $4\sqrt{3}$ and $\sqrt{3}$

Solution: $4\sqrt{3} \cdot \sqrt{3} = 4(3) = 12$ Ans.

Example 4: Multiply $\sqrt{6} + 1$ by $5\sqrt{6}$

Solution: $5\sqrt{6}(\sqrt{6} + 1) = 5(6) + 5\sqrt{6}$
 $= 30 + 5\sqrt{6}$ Ans.

Example 5: Find the square of $7 - 3\sqrt{2}$

First Solution:

$$\begin{aligned} (7 - 3\sqrt{2})^2 &= \\ 7^2 - 2 \cdot 7 \cdot 3\sqrt{2} + (3\sqrt{2})^2 &= \\ 49 - 42\sqrt{2} + 9(2) &= \\ 49 - 42\sqrt{2} + 18 &= \\ 67 - 42\sqrt{2} \end{aligned}$$

Second Solution:

Multiplying,

$$\begin{array}{r} 7 - 3\sqrt{2} \\ 7 - 3\sqrt{2} \\ \hline 49 - 21\sqrt{2} \\ - 21\sqrt{2} + 9(2) \\ \hline 49 - 42\sqrt{2} + 18 \\ \text{or} \\ 67 - 42\sqrt{2} \end{array}$$

EXERCISE 105

1. $\sqrt{2} \cdot \sqrt{8}$

8. $\sqrt[3]{9} \cdot \sqrt[3]{9}$

2. $3\sqrt{5} \cdot \sqrt{5}$

9. $\sqrt[3]{a^2} \cdot \sqrt[3]{7a}$

3. $\sqrt{a} \cdot \sqrt{ab^2}$

10. $\sqrt[3]{\frac{8}{9}} \cdot \sqrt[3]{\frac{2}{9}}$

4. $\sqrt{12} \cdot \sqrt{2}$

11. $\sqrt{3}(\sqrt{7} + \sqrt{2})$

5. $\sqrt{ax} \cdot \sqrt{\frac{1}{a}}$

12. $\sqrt{5}(\sqrt{5} - \sqrt{6})$

6. $3\sqrt{rs} \cdot \sqrt{r^3s^2}$

13. $3\sqrt{10}(2\sqrt{10} - 1)$

7. $\sqrt[3]{4} \cdot \sqrt[3]{2}$

14. $\sqrt[3]{2}(\sqrt[3]{4} + 1)$

15. $\sqrt{2}(1 + \sqrt{2} + \sqrt{3})$
16. $(5 + \sqrt{3})(5 + 2\sqrt{3})$
17. $(7 + \sqrt{10})(7 - \sqrt{10})$
18. $(a + \sqrt{b})(a - \sqrt{b})$
19. $(2\sqrt{5} - 3)(2\sqrt{5} - 3)$
20. $\left(\frac{3 + \sqrt{11}}{2}\right)\left(\frac{3 - \sqrt{11}}{2}\right)$
21. If $y = 2 - \sqrt{2}$ find the value of $y^2 + 3y$.
22. If $x = 5 + \sqrt{3}$ show that $x^2 - 10x = -22$.
23. Show that $x = 2 - \sqrt{7}$ satisfies the equation $x^2 - 4x - 3 = 0$.
24. Is $2 + \sqrt{7}$ a root of the equation in Example 23?

DIVISION OF RADICALS

An example in the division of radicals should be written in the form of a fraction. The division itself depends upon the principle that $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ and upon the methods of simplifying radicals already learned.

Example: Divide $\sqrt{32}$ by $\sqrt{8}$

Solution: $\frac{\sqrt{32}}{\sqrt{8}} = \sqrt{\frac{32}{8}} = \sqrt{4} = 2 \quad \text{Ans.}$

EXERCISE 106

Divide, and simplify:

1. $\sqrt{18} \div \sqrt{3}$

2. $\frac{\sqrt{20}}{\sqrt{5}}$

3. $2\sqrt{40} \div \sqrt{5}$

4. $\sqrt[3]{16} \div \sqrt[3]{2}$

5. $\frac{\sqrt{6}}{\sqrt{12}}$

6. $\sqrt[3]{\pi D^3} \div \sqrt[3]{\pi}$

7. $5\sqrt{\frac{8}{3}} \div \sqrt{\frac{3}{2}}$

8. $\sqrt{\frac{a}{x}} \div \sqrt{\frac{x}{a}}$

RATIONALIZING THE DENOMINATOR OF A FRACTION

Most examples in the division of radicals may be solved and computed more efficiently if we multiply the numerator and denominator by such a factor as will rid the denominator of the radical sign. This process of changing a fraction to one whose denominator is free of all radicals is known as *rationalizing the denominator*.

Example 1: Rationalize the denominator of $\frac{3}{\sqrt{2}}$

$$\text{Solution: } \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad \text{Ans.}$$

Example 2: Divide $\sqrt[3]{3}$ by $\sqrt[3]{2}$

$$\text{Solution: } \frac{\sqrt[3]{3}}{\sqrt[3]{2}} = \frac{\sqrt[3]{3}}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{\sqrt[3]{12}}{\sqrt[3]{8}} = \frac{\sqrt[3]{12}}{2} \quad \text{Ans.}$$

Example 3: In the simplest way compute the value of $\frac{1}{5\sqrt{2}}$

Solution: The simplest way is first to rationalize the denominator.

$$\begin{aligned} \frac{1}{5\sqrt{2}} &= \frac{1}{5\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{5(2)} = \frac{\sqrt{2}}{10} \\ \frac{\sqrt{2}}{10} &= \frac{1.4142}{10} = .14142 \quad \text{Ans.} \end{aligned}$$

EXERCISE 107

Rationalize the denominator in each example:

1. $\frac{5}{\sqrt{3}}$ Ans. $\frac{5\sqrt{3}}{3}$

3. $\frac{7}{\sqrt{7}}$

2. $\frac{1}{\sqrt{2}}$

4. $10 \div \sqrt{5}$

5. $\frac{2}{5\sqrt{6}}$

9. $\frac{\sqrt{a}}{\sqrt{b}}$

6. $\frac{6}{\sqrt{8}}$

10. $\frac{4}{\sqrt[3]{9}}$ Ans. $\frac{4\sqrt[3]{3}}{3}$

7. $\frac{h}{\sqrt{3}}$

11. $\frac{1}{\sqrt[3]{m}}$

8. $\frac{1}{\sqrt{a}}$

12. $\frac{R}{\sqrt[3]{R^2}}$

13. (a) Divide 5 by 1.414, to four significant figures.

(b) Rationalize the denominator of $\frac{5}{\sqrt{2}}$ and find its value as in Model Example 3. Compare the value with (a).

After rationalizing the denominator of each fraction below, simplify and find the value of the result to four figures, with the help of the tables:

14. $\frac{1}{2\sqrt{2}}$ Ans. $\frac{\sqrt{2}}{4} = .3536$

19. $\frac{7}{\sqrt{10}}$

15. $\frac{\sqrt{5}}{\sqrt{2}}$

20. $\frac{3}{\sqrt[3]{3}}$

16. $\frac{3}{\sqrt{3}}$

21. $\frac{10}{\sqrt[3]{9}}$

17. $\frac{2}{\sqrt{5}}$

22. $\frac{1}{\sqrt[3]{2}}$

18. $\frac{1}{2\sqrt{5}}$

23. $\frac{10}{\sqrt[3]{100}}$

24. Divide $2h$ by $\sqrt{3}$ and find its value if $h = 100$.

25. Divide $\frac{d}{\sqrt{2}}$ and find its value if $d = 12$.

THE USE OF SQUARE ROOT TABLES

The process of square root takes a good deal of time, particularly when the result is properly checked by multiplication. Practical men nowadays depend more and more upon tables as printed in their handbooks, and it is an essential part of one's education to be able to understand the construction and use of tables.

Tables of square roots of numbers from $\widehat{1.00}$ to $\widehat{9.99}$ are printed in ten columns on pages 378 and 379. Look for the first two figures of the given number at the left of the page, the third figure at the top. To find the square root of 3.56 locate 3.5 at the left; then go across the page until under the column headed 6 you see 1.887; therefore $\sqrt{3.56} = 1.887$. It is a good plan to make an advance estimate of the square root in round numbers, and also to set the number off in periods of two figures each so as to be sure to look on the right page.

The square root of $\widehat{35.60}$ is found on the page headed "square roots of numbers from $\widehat{10.00}$ to $\widehat{99.90}$." It is opposite 35 and under 6, where you will find 5.967. If doubtful about any answer, test it by multiplication or by using the square root process.

If the number were 3560 or .356 or .00356 the significant figures of the square root would be the same as for 35.60, namely 5967, but the decimal point would not be in the same place. In the case of $\widehat{3560}$, there are two periods before the decimal point and the root would be 59.67. What would be the square root of .3560? of $\widehat{.003560}$? One can usually place the decimal point correctly by a common-sense estimate.

Follow these directions:

1. Set off the given number in periods.
2. Make a rough estimate of the square root.

3. (a) If there is only one figure in the first period, *i.e.* if the first period is between $\widehat{01}$ and $\widehat{09}$, turn to the pages headed $\widehat{1.00}$ to $\widehat{9.99}$.

(b) If there are two figures in the first period, *i.e.*, if the first period is between $\widehat{10}$ and $\widehat{99}$, turn to the pages headed $\widehat{10.00}$ to $\widehat{99.90}$.

4. Locate the square root on the proper page, line and column.

5. Place the decimal point in the root according to your estimate, or according to the number of periods before or after the decimal point in the number.

6. Consider finally whether the answer is a reasonable one, and whether it agrees with your rough estimate.

Example 1: Find the square root of .31.

(1) Set off in periods $\widehat{.31}$.

(2) A rough estimate, since $\widehat{.31}$ is about midway between .25 and .36, would be $.5\frac{1}{2}$ or say .55.

(3) Since there are two figures in the first period $\widehat{31}$ we look on the pages headed $\widehat{10.00}$ to $\widehat{99.90}$.

(4) Find 31 at the left, and opposite 31 under 0 we find 5.568.

(5) Place the decimal point in accordance with our estimate, and with the lack of any periods before the decimal point. The result is .5568.

Example 2: Find the square root of 31000.

(1) Set off in periods $\widehat{31000}$.

(2) A rough estimate; something more than 170, since $\widehat{31000}$ is more than $\widehat{28900}$.

(3) Since there is only one figure in the first period we look on the pages headed $\widehat{1.00}$ to $\widehat{9.99}$.

(4) Find 31 at the left, and opposite 31 under 0 we find 1.761.

Since 1.665 is halfway between 1.66 and 1.67 its square root is halfway between 1.288 and 1.292, which gives us 1.290,

$$\therefore \sqrt{1.665} = 1.290$$

Find in the same way $\sqrt{1.815}$, $\sqrt{33.65}$, and $\sqrt{6485}$.

Commonly the interpolating is based upon tenths, as follows:

Example 2: Find the square root of 24.63.

From the tables: $\sqrt{24.60} = 4.960$

$$\sqrt{24.70} = 4.970$$

24.63 is $\frac{3}{10}$ of the way from 24.60 to 24.70, therefore its square root is $\frac{3}{10}$ of the way from 4.960 to 4.970.

The difference between 4.960 and 4.970 is .010 (the "tabular difference"). $\frac{3}{10}$ of .010 is .003.

Adding .003 (the correction) we find that the square root of 24.63 is 4.963.

Example 3: Find the square root of 10.16.

$$\sqrt{10.20} = 3.194$$

$$\sqrt{10.10} = 3.178$$

$$.016 = \text{the tabular difference}$$

Since 10.16 is $\frac{6}{10}$ of the way from 10.10 to 10.20, its square root is $\frac{6}{10}$ of the way from 3.178 to 3.194.

The correction is $\frac{6}{10}$ of .016 = $\frac{.096}{10} = .0096$ or to the nearest thousandth .010.

Adding the correction .010, the square root of 10.16 = 3.188

Example 4: Find the square root of 20.87.

$$\sqrt{20.90} = 4.572$$

$$\sqrt{20.80} = 4.561$$

$$.011 = \text{the tabular difference}$$

The correction is $\frac{7}{10}$ of .011 = $\frac{.077}{10} = .0077$ or .008 -

Adding .008 we get the square root of 20.87 = 4.569.

The result of any example can be checked by obtaining the square root by the regular process, by multiplying, or by dividing.

EXERCISE 109

Using the tables find the square root of each number to four figures:

- | | |
|---|------------|
| 1. 12.65 (correction is $\frac{5}{10}$ of .014) | |
| 2. 12.67 (correction is $\frac{7}{10}$ of — = .010) | |
| 3. 24.68 | 13. 25.63 |
| 4. 1013 | 14. 10.25 |
| 5. 1.662 | 15. 1.781 |
| 6. 9.225 | 16. 2.999 |
| 7. 1.034 | 17. .03122 |
| 8. 107.9 | 18. .7854 |
| 9. 2134 | 19. 39.37 |
| 10. 1925 | 20. 11110 |
| 11. 314.2 | 21. 43560 |
| 12. 246.6 | 22. 564700 |

Five-figure or six-figure numbers should first be rounded off to four-figure approximations. Instead of $\sqrt{42516}$ find it as $\sqrt{42520}$; instead of $\sqrt{1.2364}$, find it as $\sqrt{1.236}$. These tables cannot be used to obtain results correct to more than four significant figures.

- | | |
|------------|-------------|
| 23. 4273.2 | 26. 31948 |
| 24. 129.34 | 27. 428675 |
| 25. 2344.7 | 28. 3.14159 |

FRACTIONAL EXPONENTS

To find a meaning for such an expression as $a^{\frac{1}{2}}$ we shall assume that our old rule for the multiplication of powers of the same base holds true when the exponents are fractional. Then $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$ will be equal to $a^{\frac{1}{2} + \frac{1}{2}}$ or a . Now if a number

is the product of two equal factors, each of the factors is called the square root of the number. Therefore since a is the product of the two equal factors $a^{\frac{1}{2}}$ and $a^{\frac{1}{2}}$, $a^{\frac{1}{2}}$ is the square root of a .

If $a^{\frac{1}{2}} = \sqrt{a}$ it is clear that the fractional exponent $\frac{1}{2}$ indicates that the square root of the base is to be taken.

In a similar way we find the meaning of any other fractional exponent.

Example 1: To find the meaning of $a^{\frac{1}{3}}$.

$$\begin{aligned} \text{Solution: } a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} &= a^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\ &= a^{\frac{3}{3}} \\ &= a \end{aligned}$$

Therefore $a^{\frac{1}{3}} = \sqrt[3]{a}$. Ans.

What is the meaning of $x^{\frac{1}{4}}$? $y^{\frac{1}{5}}$?

Example 2: To find the meaning of $x^{\frac{3}{4}}$.

$$\begin{aligned} \text{Solution: } x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{4}} &= x^{\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}} \\ &= x^{\frac{12}{4}} \\ &= x^3 \end{aligned}$$

Therefore $x^{\frac{3}{4}} = \sqrt[4]{x^3}$, since $x^{\frac{3}{4}}$ is one of the four equal factors which give the product x^3 . Ans.

In general, the denominator of a fractional exponent indicates what root of the base is to be taken, and the numerator indicates to what power the base is to be raised.

Example 3: What is the value of $8^{\frac{2}{3}}$?

$$\text{Solution: } 8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4. \quad \text{Ans.}$$

ORAL EXERCISE

Write the following expressions as radicals:

1. $m^{\frac{2}{3}}$

5. $3^{\frac{1}{2}}$

2. $a^{\frac{3}{5}}$

6. $4^{\frac{1}{3}}$

3. $x^{\frac{3}{2}}$

7. $2^{\frac{3}{4}}$

4. $y^{\frac{4}{3}}$

8. $5^{\frac{3}{2}}$

Write each of the following with fractional exponents:

- | | |
|---------------------|---------------------|
| 9. $\sqrt{b}$ | 13. $\sqrt{5}$ |
| 10. $\sqrt[3]{a}$ | 14. $\sqrt[3]{3}$ |
| 11. $\sqrt{x^3}$ | 15. $\sqrt[4]{2^2}$ |
| 12. $\sqrt[3]{x^2}$ | 16. $\sqrt[3]{3^5}$ |

Find the value of each of the following expressions:

- | | |
|--|-------------------------------------|
| 17. $49^{\frac{1}{2}}, 100^{\frac{1}{2}}, 1^{\frac{1}{2}}$ | 22. $(-64)^{\frac{1}{3}}$ |
| 18. $27^{\frac{1}{3}}, 1000^{\frac{1}{3}}$ | 23. $(\frac{1}{16})^{\frac{1}{2}}$ |
| 19. $4^{\frac{3}{2}}$ | 24. $(\frac{1}{6^4})^{\frac{1}{3}}$ |
| 20. $8^{\frac{2}{3}}$ | 25. $81^{\frac{3}{4}}$ |
| 21. $32^{\frac{3}{5}}$ | 26. $(-125)^{\frac{2}{3}}$ |
27. $7225^{\frac{1}{2}}$ Refer to the first page of the tables.
28. $2^{\frac{1}{2}}$ Refer to the first page of the tables.
29. $(x^2 + 2xy + y^2)^{\frac{1}{2}}$
30. $(4a^2 - 4a + 1)^{\frac{1}{2}}$

Using the rule for the multiplication of powers of the same base, express each of the following in a simpler form.

31. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$
32. $b^{\frac{2}{3}} \cdot b^{\frac{3}{2}}$
33. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{3}{4}}$
34. $2 \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{5}}$

ZERO AND NEGATIVE EXPONENTS

By a method similar to that used in finding the meaning of a fractional exponent, we can find meanings for zero and negative exponents.

Example 1: Find the meaning of a^0

Solution: Assuming that the rule for the multiplication of powers of the same base holds true $a^0 \cdot a^5 = a^{0+5}$

$$a^0 \cdot a^5 = a^5$$

Dividing each member by a^5 ,

$$a^0 = \frac{a^5}{a^5}$$

$$a^0 = 1$$

Thus any number (except zero) with the exponent zero has the value 1.

Example 2: Find the meaning of a^{-4} .

Solution: $a^{-4} \cdot a^4 = a^{-4+4}$

$$a^{-4} \cdot a^4 = a^0$$

$$a^{-4} \cdot a^4 = 1$$

Dividing each member by a^4 ,

$$a^{-4} = \frac{1}{a^4}$$

In like manner we may find the meaning of any negative exponent.

Oral Exercise

Give the value of

1. x^0

6. 2^0

2. $x^0 + y^0$

7. $(\frac{1}{2})^0$

3. $(x + y)^0$

8. $4 \cdot 3^0$

4. $(xy)^0$

9. $4^0 \cdot 3^0$

5. $(\frac{1}{x})^0$

10. $\frac{2}{3^0}$

Find as in Model Example 2 the meaning of:

11. x^{-2}

12. a^{-5}

Find the value of each of the following:

13. 3^{-2}

18. $(-5)^{-2}$

14. 4^{-3}

19. $2^{-1} \cdot 3^{-1}$

15. 3^{-4}

20. $4 \cdot 2^{-2}$

16. $(-2)^{-3}$

21. $3^{-3} \cdot 9$

17. 4^{-1}

22. Fill in the blanks:

$$10^4 = 10,000$$

$$10^3 =$$

$$10^2 =$$

$$10^1 =$$

$$10^0 =$$

$$10^{-1} =$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = .01$$

$$10^{-3} =$$

$$10^{-4} =$$

23. Express 0.07 as the product of a whole number and a power of 10.

$$\begin{aligned} \text{Solution: } 0.07 &= 7 \times .01 \\ &= 7 \times \frac{1}{100} \\ &= 7 \times \frac{1}{10^2} \\ &= 7 \times 10^{-2} \quad \text{Ans.} \end{aligned}$$

Express each of the following as the product of a whole number and a power of 10:

24. 700

28. .17

25. 50,000

29. .015

26. 0.3

30. 2,000,000

27. 0.004

31. 0.0009

EQUATIONS SOLVED BY SQUARE ROOT

Knowledge of square root and of radicals enables us to solve many equations and formulas which involve the square of the unknown quantity.

An interesting and novel feature of such equations is that there are usually two distinct answers. If the equation is $x^2 = 25$, the x^2 is like an interrogation, asking us, "What number multiplied by itself makes 25?" It happens that there are two numbers of which this is true, $+5$ and -5 .

What other number would have the same square as -7 ? as $-3a$? as $a - b$? It may be said that any number n has two square roots, which may be indicated by $+\sqrt{n}$ and $-\sqrt{n}$.

Example 1: Solve for x , and check: $2x^2 - 7 = 425 - x^2$

Solution:

$$2x^2 - 7 = 425 - x^2$$

$$2x^2 + x^2 = 425 + 7$$

$$3x^2 = 432$$

$$x^2 = 144$$

$$x = +12 \text{ or } -12 \quad \text{Ans.}$$

Check: $x = +12$

Check: $x = -12$

Left Member	Right Member	Left Member	Right Member
$2(+12)^2 - 7 =$	$425 - (+12)^2 =$	$2(-12)^2 - 7 =$	$425 - (-12)^2 =$
$2(144) - 7 =$	$425 - 144 =$	$2(144) - 7 =$	$425 - 144 =$
$288 - 7 =$	$281 =$	$288 - 7 =$	$281 =$
$281 \longleftrightarrow 281$		$281 \longleftrightarrow 281$	

Example 2: Solve for n , and check: $n + 7 = \frac{15}{n - 7}$

Solution: $n + 7 = \frac{15}{n - 7}$

$$n^2 - 49 = 15$$

$$n^2 = 15 + 49$$

$$n^2 = 64$$

$$n = +8 \text{ or } -8$$

(These answers are sometimes written $n = \pm 8$.)

Check: $n = +8$

Check: $n = -8$

Left Member	Right Member	Left Member	Right Member
$8 + 7 =$	$\frac{15}{8 - 7} =$	$-8 + 7 =$	$\frac{15}{-8 - 7} =$
$15 =$	$\frac{15}{1} =$	$-1 =$	$-\frac{15}{15} =$
$15 \longleftrightarrow 15$		$-1 \longleftrightarrow -1$	

Example 3: Solve for x , and check: $ax^2 + b = c$

Solution: $ax^2 + b = c$

$$ax^2 = c - b$$

$$x^2 = \frac{c - b}{a}$$

$$x = +\sqrt{\frac{c - b}{a}} \text{ or } -\sqrt{\frac{c - b}{a}} \quad \text{Ans.}$$

$$\text{Check: } x = +\sqrt{\frac{c - b}{a}}$$

$$\text{Check: } x = -\sqrt{\frac{c - b}{a}}$$

$$\begin{array}{lcl} a\left(+\sqrt{\frac{c - b}{a}}\right)^2 + b = & \left| \begin{array}{c} c \\ c \\ c \end{array} \right| & a\left(-\sqrt{\frac{c - b}{a}}\right)^2 + b = \left| \begin{array}{c} c \\ c \\ c \end{array} \right| \\ a\left(\frac{c - b}{a}\right) + b = & \left| \begin{array}{c} c \\ c \\ c \end{array} \right| & a\left(\frac{c - b}{a}\right) + b = \left| \begin{array}{c} c \\ c \\ c \end{array} \right| \\ c - b + b = & \left| \begin{array}{c} c \\ c \\ c \end{array} \right| & c - b + b = \left| \begin{array}{c} c \\ c \\ c \end{array} \right| \\ c \longleftrightarrow c & & c \longleftrightarrow c \end{array}$$

EXERCISE 110

Solve, and check each answer separately:

1. $5x^2 + 9 = 729$

2. $(3w + 1)(3w - 1) = 80$

3. $(17 - x)(17 + x) - 225 = 0$

4. $(2n)^2 + (3n)^2 + (4n)^2 = 725$

5. $3\frac{1}{7}R^2 = 154$

6. $3.14D^2 = 160$

Solution: $D^2 = 160 \div 3.14$

$$D^2 = 50.96$$

$$D = \pm 7.13 \quad \text{Use the tables of square roots.}$$

7. $3.14r^2 = 144$

8. $0.015l^2 = 62.4$

9. $9 : x = x : 25$

10. $\frac{35r}{6} = \frac{210}{r}$

$$11. \frac{3 - 5x}{x + 2} = \frac{1 - 7x}{5 + 4x}$$

$$12. \frac{y + 5}{y - 5} + \frac{y + 5}{y - 5} = \frac{58}{y^2 - 25}$$

$$13. \frac{n}{n - 2} = \frac{18}{5} - \frac{n}{n + 2}$$

$$14. \text{Solve for } t \text{ if } g = 32: \quad \frac{1}{2}gt^2 = 400$$

$$15. \text{Solve for } R: \quad A = \pi R^2$$

$$16. \text{Solve for } t: \quad s = \frac{1}{2}gt^2$$

$$17. \text{Change the subject of the formula } E = \frac{1}{2}mv^2 \text{ to } v.$$

$$18. \text{Change the subject of the formula } S = 4\pi R^2 \text{ to } R.$$

$$19. \text{Solve for } x: \quad \left(\frac{a}{2}\right)^2 + x^2 = \frac{5a^2}{2}$$

$$20. \text{Solve for } R: \quad V = \pi R^2 h$$

$$21. \text{Solve for } h: \quad d^2 = l^2 + w^2 + h^2$$

In the following examples use the tables of square roots where necessary:

22. Find the radius of a circle the area of which is 160 square rods, to three figures.

23. Find the radius and diameter in inches of a circular disc covering one square foot.

24. Find the radius of a cylindrical can containing one gallon (231 cu. in.), the height being 10".

THE SIDES OF A RIGHT TRIANGLE

EXERCISE 111

1. (a) Construct with ruler and compasses a right triangle having one of its arms (the arms are the perpendicular sides) $1\frac{1}{2}$ " long and the other 2" long.

(b) Construct squares on the two arms and on the third side, the "hypotenuse."

(c) Divide each of the three squares into smaller squares, each $\frac{1}{2}$ " on a side.

(d) Count the number of $\frac{1}{2}$ " squares in the square on each of the three sides. How does the size of the square on the hypotenuse compare with the size of the other two squares combined?

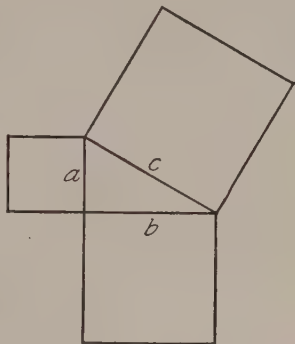
2. Construct carefully a right triangle having its arms 6" and $2\frac{1}{2}$ " long. Measure the hypotenuse, square its length, and compare with the sum of the squares of 6 and $2\frac{1}{2}$.

3. Copy this statement, and complete it: The square on the — of a — — is equal to the sum of the — on the two —.

The fact which you have just stated was first proved by Pythagoras, a wise Greek philosopher who lived over 2000 years ago. It may be expressed by the formula

$$c^2 = a^2 + b^2$$

The formula is illustrated in the diagram opposite. Can you solve it for c ?

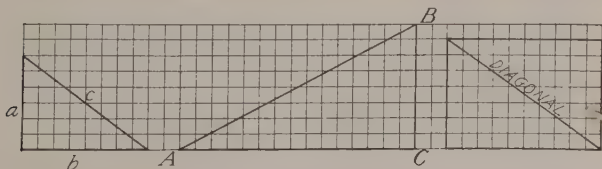


4. Find a formula for a in terms of the other two sides of the triangle.

5. Find a similar formula for b .

6. If in the diagram a were equal to 5 centimeters and b to 12 centimeters, what would be the length of c ?

7. If in the diagrams below each square represents a



square inch, how long is b ? a ? Calculate c by the formula. Check by measuring the actual length c .

8. What length is represented by AC ? by BC ? Calculate AB and check the calculation by measurement.

9. Calculate the diagonal of the rectangle after observing the length and width. Use the square root tables whenever they can help you. Check by measuring.

10. The base of a right triangle is $15'$ and the hypotenuse is $39'$. Find the altitude.

11. One of the arms of a right triangle is $28''$ and the hypotenuse is $53''$. Find the other arm.

Find the missing side of each right triangle, leaving radical answers in simplest form without computing:

	BASE b	ALTITUDE h	HYPOTENUSE c
12.	$3''$	$4''$	
13.		$14'$	$52'$
14.	a	a	
15.	$10''$		$20''$
16.	5	$5\sqrt{3}$	
17.	$7\sqrt{2}$	$7\sqrt{2}$	
18.	$p^2 - q^2$	$2pq$	
19.	$3n$	$4n$	

20. A triangle the sides of which are proportional to 3, 4 and 5 (see Examples 7, 12, and 19) is often used by surveyors and mechanics to obtain right angles. Explain how a carpenter could measure on a level floor $8'$ out from the foot of an upright he is to place in position, and use a 10 ft. pole to establish a true right angle. Sketch.

21. Can you plan how a surveyor and his assistants might use $15'$, $20'$, and ? lengths to lay out a right angle on flat ground? If possible to secure a 50 ft. tape try it yourself.

22. Can a $36''$ umbrella be laid flat on the bottom of a trunk $30'' \times 18''$?

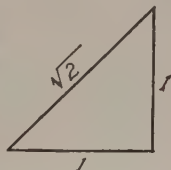
23. A baseball diamond is a square 90' on each side. How far is it from home plate to second base? from third base to first base?

24. Find to the nearest foot how much is saved by cutting diagonally across a vacant corner lot $100' \times 50'$, instead of walking along the two sides of the lot.

25. If Newark is ten miles west of City Hall, New York, and Yonkers is fifteen miles north, how far is it from Newark to Yonkers "as the crow flies"?

26. With compasses and ruler make a drawing for the preceding example, scale $\frac{1}{4}" = 1$ mile. Check No. 25 by measurement.

27. The hypotenuse of this triangle represents graphically $\sqrt{2}$. Construct with instruments two perpendicular lines and form a right triangle with arms 2" and 1". What radical is represented by the hypotenuse?



28. Construct in a similar way a right triangle the hypotenuse of which will represent $\sqrt{10}$. How long will you make the arms?

29. Draw a circle, radius 1". Construct two diameters perpendicular to each other and join their ends. This forms a square which is said to be inscribed in the circle. Calculate the length of its side, which will be the hypotenuse of a right triangle. Check your calculation roughly by measuring the side.

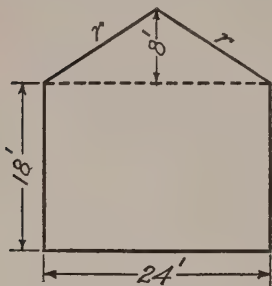
30. Inscribe a square in a circle of radius $1\frac{1}{2}"$, and calculate its side as before.

31. Find a formula for the side of an inscribed square in terms of the radius of the circle (R). (Answer $R\sqrt{2}$.)

32. If the side of an inscribed square is s , find a formula for the radius of the circle; for the diameter of the circle.

33. With compasses construct an isosceles triangle, base 3" long, arms each $2\frac{1}{2}"$ long. Let fall the altitude of the

triangle and measure it. Calculate the altitude by the formula of Pythagoras and check. (The altitude bisects the base.)



34. The gable end of a house is illustrated. Find its area in square feet, and find the length of the sloping rafter r .

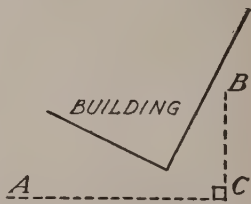
35. If the base of an isosceles triangle is b and each arm is a find a formula for the altitude h .

36. With compasses construct an equilateral triangle, side 2". Let fall the altitude and measure it. Calculate it by the formula of Pythagoras and check.

37. If the side of an equilateral triangle is 10" find the altitude and the area of the triangle.

38. If the side of an equilateral triangle is a , find a formula for the altitude.

39. Because of an intervening building it is impossible to measure directly from A to B . By making a right angle at C , how could the builder calculate AB ? How long is AB if $AC = 22'$, $BC = 15'$?

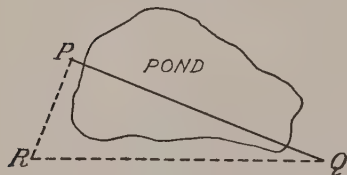


40. If $a^2 + b^2 = c^2$ then $a^2 = ?$

Factor the right member and use the resulting formula to calculate a , if

(a) $c = 50'$, $b = 48'$.

(b) $c = 1000'$, $b = 995'$.



41. P and Q are on opposite sides of a pond. To find PQ a perpendicular was constructed at P ; PR was measured, 150.2'; RQ was found to be 228.1'. Calculate PQ , using the method of Example 40.

42. Find a formula for the diagonal of a square in terms of a side.

43. If the diagonal of a square is $5''$ show that the side is $\frac{5}{2}\sqrt{2}$. What is the area of the square?

44. Find the side and area of the square if the diagonal is d inches.

EXERCISE 112

Solve each of these problems by using an equation:

1. A rectangle is three times as long as it is wide, and its area is 192 square inches. Find the length and width.

2. One arm of a right triangle is double the other and the hypotenuse is $\sqrt{80}$. What are the arms?

3. Two numbers are in the ratio $5 : 3$ and their product is 960. What are the numbers?

4. Find two numbers in the ratio $2 : 7$ the difference of whose squares is 405.

5. Find a number which is four times its reciprocal.

6. A rectangular solid is $5''$ high, and twice as long as it is wide. Find its width if the volume is 360 cubic inches.

7. The total surface of a cube is 96 square inches. What is the edge of the cube?

8. The hypotenuse of an isosceles right triangle is $\sqrt{242}$. How long are the arms?

9. The length and width of a rectangle are proportional to $12 : 5$ and the diagonal is 65 feet. Find the length and width.

10. Find three consecutive numbers the sum of whose squares is 110. (Hint: Let x = the middle number.)

11. The product of two consecutive numbers exceeds the smaller of the two numbers by 25. What are the numbers?

CHAPTER XIII

NUMERICAL TRIGONOMETRY

THE TANGENT OF AN ANGLE

EXERCISE 113

1. On a piece of 10×10 cross-section paper draw a horizontal line AC 5 inches long. At A construct very carefully with your protractor an angle of 35° . At C draw a line perpendicular to AC and meeting the other side of the angle A at B , forming a right triangle ABC . What is the length of CB ? What is the value of the ratio of CB to AC ? Express as a decimal.

2. In the same diagram locate on AC a point 2 inches away from A . Call this point C_1 . At C_1 erect a perpendicular to AC and extend it until it meets the line AB at B_1 . In the right triangle AC_1B_1 what is the length of the side B_1C_1 ? Express as a decimal the ratio of C_1B_1 to AC_1 .

3. Repeat Example 2, making the point C_2 four inches away from A . Compare the ratio $\frac{B_2C_2}{AC_2}$ in the triangle AC_2B_2 with the ratios $\frac{C_1B_1}{AC_1}$ and $\frac{CB}{AC}$ obtained in the previous examples.

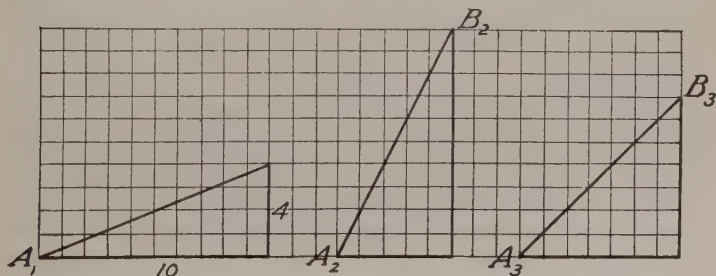
4. The three triangles in the preceding examples are similar triangles. Why? What is true of the ratios of corresponding sides of similar triangles? How then should the ratio of CB to AC compare in value with the ratio C_1B_1 to AC_1 in the second triangle? with the ratio B_2C_2 to AC_2 ?

From these examples we learn that in any right triangle which contains an angle of 35° the ratio of the side opposite

the angle to the side adjacent to the angle is always the same, and that its value is about 0.7.

This ratio of the side opposite an acute angle in a right triangle to the side adjacent to the angle is used so much that it has received a name; it is called the *tangent of the angle*. Thus in Example 1 the ratio of CB to AC is the tangent of angle A and we write, $\text{tangent } A = \frac{CB}{AC}$, or in abbreviated form $\tan A = \frac{CB}{AC}$.

5. In the diagram, what is the ratio of the side opposite the angle A_1 , to the side adjacent to the angle? Observe the



sides of the other two triangles and give the value of $\tan A_2$ and of $\tan A_3$.

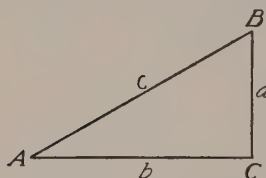
6. Find in the same way in which you found a value for $\tan 35^\circ$, a value for $\tan 60^\circ$. Use at least two different triangles.

7. In the diagram for Example 5, find the tangent of the angle B_2 ; find also $\tan B_3$.

8. With your protractor measure the angle A_2 , and thus find how many degrees there are in an angle whose tangent is 2.

By drawing diagrams and measuring we might approximate the value of the tangent of any acute angle. Since so

much depends upon the carefulness with which the diagram is constructed mathematicians sought and found other and better ways of obtaining these values. The methods which they found bring accurate results, and have given us complete tables of the tangent of any angle. (See pages 382–383.)



In problems dealing with a right triangle it is customary to letter it as in the accompanying diagram. C is always the right angle, c the hypotenuse, and a and b opposite the angles A and B respectively.

9. What is the value of $\tan A$ if $a = 45''$ and $b = 60''$?
10. Find $\tan B$ if $a = 8'$ and $b = 6'$.
11. What is $\tan A$ if $b = .40$ and $a = .09$?
12. If $a = 5''$ and $c = 13''$, how can you find, without measuring, the length of b ? Find b and then find $\tan A$.
13. What is $\tan B$ if $a = 8'$ and $c = 17'$?
14. $c = 1.3$, $b = 1.2$, what is the value of $\tan A$?
15. Write the formula for $\tan A$ in terms of a and b . Solve it for b , and for a .
16. Write the formula for $\tan B$ and change the subject of the formula to a .

USE OF THE TANGENT

Many problems involving heights and distances can readily be solved by the use of the tangent. Without a knowledge of this method we should be obliged to use the rather cumbersome and inaccurate method of scale drawing.

Example: A telephone pole is anchored by a wire from the top. This wire reaches the ground $18'$ from the foot of the pole and makes an angle of 58° with the ground. What is the height of the pole? Given that $\tan 58^\circ = 1.60$.

First Solution: Make on squared paper a diagram to scale and read off the desired length. This result will not be accurate but will serve as a rough check.

Second Solution: Represent by x the height of the pole.

$$\text{Then, from the diagram } \tan 58^\circ = \frac{x}{18}$$



We are given that $\tan 58^\circ = 1.60$; substituting for $\tan 58^\circ$ this value, we have $1.60 = \frac{x}{18}$

$$\therefore x = 18(1.60)$$

$x = 28.8$, the height of the pole.

EXERCISE 114

1. What is the length of a in the right triangle ABC if $A = 37^\circ$ and $b = 20'$? The value of $\tan A$ is 0.75.

2. Find b if $A = 64^\circ$ and $a = 61.5''$. ($\tan 64^\circ = 2.05$.)

3. If $A = 23^\circ$ and $a = 190'$, what is the length of b if $\tan 23^\circ = 0.42$?

4. A surveyor had to find the distance from B to C . He could not measure this directly because a small pond intervened. So he measured CA making a right angle with CB ,

500' in length, and with the transit (an instrument used for measuring angles)



found the angle at A to be 72° . What is the distance from C to B ? ($\tan 72^\circ = 3.078$.)

5. A tree 50' high stands on the bank of a small river. From a point on the other bank directly opposite, the line

of sight to the top of the tree makes an angle of 12° with the horizontal. Find the number of feet across the river. ($\tan 12^\circ = 0.213$.)

Use of a Table of Tangents

In the examples which you have just been working the value of the tangent of the angle was given in each problem. Usually only the value of the angle is given and the tangent must be obtained from a table. Such a table you will find on pages 382-383. In the table the values of the tangents of all the angles from 0° to 90° at $10'$ (10-minute) intervals are given to four decimal places. This table is used in very much the same way as the square root tables. The number of degrees in the angle is to be found at the left of the page; the number of minutes is at the top. A few examples will make the use of the table clear to you.

Example: What is the value of $\tan 38^\circ 10'$?

Solution: Locate 38° at the left of the page, and opposite 38° in the column headed $10'$ you will find .7860.

$$\therefore \tan 38^\circ 10' = 0.7860 \quad \text{Ans.}$$

EXERCISE 115

1. Using the table find:

- | | |
|-------------------------|-------------------------|
| (a) $\tan 18^\circ 10'$ | (d) $\tan 43^\circ 20'$ |
| (b) $\tan 27^\circ 30'$ | (e) $\tan 2^\circ 40'$ |
| (c) $\tan 34^\circ 50'$ | (f) $\tan 33^\circ 10'$ |

2. Find the values of the following:

- | | |
|-------------------------|-------------------------|
| (a) $\tan 55^\circ$ | (d) $\tan 83^\circ 10'$ |
| (b) $\tan 60^\circ 20'$ | (e) $\tan 46^\circ 30'$ |
| (c) $\tan 72^\circ 40'$ | (f) $\tan 66^\circ 50'$ |

3. What is the value of angle A in degrees and minutes if $\tan A = 0.2742$?

Solution: We hunt through the table until we locate the number 0.2742.

We find that it lies opposite 15° and in the column under the heading $20'$.

$$\therefore 0.2742 = \tan 15^\circ 20'$$

4. Find angle A in degrees and minutes if (a) $\tan A = 0.5969$; (b) $\tan A = 0.8796$; (c) $\tan A = 1.8291$.

5. In a right triangle ABC find the length of a if $b = 200'$ and angle $A = 29^\circ 10'$.

6. If $a = 152'$ and $A = 37^\circ 30'$, what is the length of b ?

7. A boy wishing to find the height of a flag pole, measured off on level ground $100'$ from the base of the pole and with an instrument for measuring angles found the angle between the horizontal and the line of sight to the top of the pole to be $43^\circ 20'$. How high was the flag pole?

If the angle is not found in the tables we must use interpolation just as we did in the square root tables.

Example: Find $\tan 32^\circ 18'$.

Solution: The angle $32^\circ 18'$ lies between $32^\circ 10'$ and $32^\circ 20'$.

From the tables,

$$\tan 32^\circ 20' = .6330$$

$$\tan 32^\circ 10' = .6289$$

$$\text{Difference} = .0041$$

Now if we are seeking $\tan 32^\circ 18'$, then to $\tan 32^\circ 10'$ we add $\frac{8}{10}$ of the difference.

$$\frac{8}{10} \text{ of } .0041 = .0033$$

$$\therefore \tan 32^\circ 18' = 0.6289 + .0033$$

$$= 0.6322 \quad \text{Ans.}$$

EXERCISE 116

1. Show that the value of $\tan 27^\circ 25'$ is .5188.

2. Find $\tan 15^\circ 45'$.

3. If $A = 56^\circ 25'$, what is the value of $\tan A$?

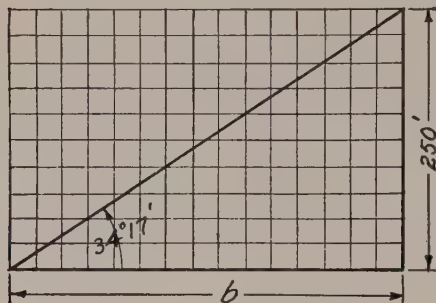
4. Using interpolation find:

(a) $\tan 10^\circ 22'$

(c) $\tan 62^\circ 33'$

(b) $\tan 31^\circ 38'$

(d) $\tan 64^\circ 59'$



5. In the right triangle ABC , $b = 150'$, $A = 44^\circ 36'$. What is the length of a ?

6. In the right triangle in the diagram, what is the length of b ?

7. The base of a rectangle makes with the diagonal an angle of $22^\circ 42'$. If the base is $18''$ long,

what is the length of the altitude of the rectangle?

8. What is the angle whose tangent is .6598?

Solution: We cannot find 0.6598 in the table of tangents, so we find the two values between which it lies, 0.6577 and 0.6619.

$$0.6577 = \tan 33^\circ 20'$$

$$0.6619 = \tan 33^\circ 30'$$

$$.0042 = \text{increase in value of tangent for an increase of } 10'$$

An increase of .0042 in the value of the tangent makes an increase of $10'$ in the angle, therefore an increase of .0021 (the difference between 0.6577 and 0.6598) makes an increase in the angle of $\frac{.0021}{.0042}$ of $10'$, or $\frac{1}{2}$ of $10'$, which is $5'$.

$$\therefore 0.6598 = \tan 33^\circ 25'$$

9. What is the angle whose tangent is 0.6068?

10. $\tan A = 0.7545$. Find A in degrees and minutes.

11. Find angle B in each case:

(a) $\tan B = .1740$

(c) $\tan B = 1.6191$

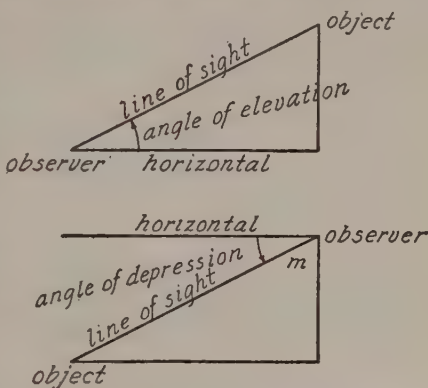
(b) $\tan B = .7463$

(d) $\tan B = 2.2424$

12. In the right triangle ABC , $a = 172.1'$ and $b = 245'$; find angle A .

13. The base of a rectangle is $24''$ and the altitude $10''$. What angle does the diagonal of the rectangle make with the base?

If we are sighting an object above the horizontal plane, the angle between the horizontal and the line of sight is called the *angle of elevation* of the object. If the object is below the horizontal plane, the angle between the horizontal and the line of sight is called the *angle of depression* of the object.



Note that in Diagram 2 the angle m can be found by subtracting the number of degrees in the angle of depression from 90° .

In all examples, first construct to scale on squared paper a diagram as clear and as accurate as possible, and then scale off from the diagram the result desired. This will serve as a rough check on the computation.

EXERCISE 117

1. At a distance of $150'$ from the foot of a tower the angle of elevation of the top of the tower is $42^\circ 30'$. What is the height of the tower?

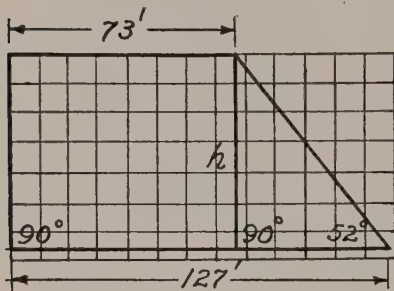
2. What is the height of a church tower, if at a point $70'$ from its base the angle of elevation of the top is $73^\circ 40'$?

3. What is the angle of elevation of the sun if a pole $37'$ high casts a shadow of $50'$?

4. A flagpole 35' high stands at the edge of a small pond. From a point on the opposite shore the angle of elevation of the top is 10° . How far is it across the pond?

5. From the top of a lighthouse 160' high the angle of depression of a floating buoy is $22^\circ 15'$. How far is the buoy from the foot of the lighthouse, if the lighthouse is at the water's edge?

6. The base of a rectangle is 32' long. What is the alti-



tude of the rectangle if the diagonal makes an angle of $27^\circ 13'$ with the base?

7. In the annexed diagram, find

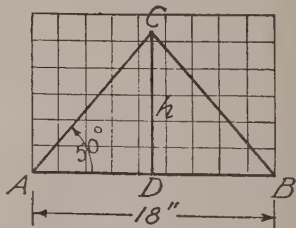
(a) h , the altitude of the trapezoid;

(b) the area of the trapezoid.

8. The length and the width of a rectangle are 135' and 76' respectively. What angle does the diagonal make with the long side?

9. When the angle of elevation of the sun is $54^\circ 17'$, what is the height of a tree which casts a shadow 63' in length?

10. ABC is an isosceles triangle. CD is perpendicular to AB and bisects AB . Find the length of CD and the area of the triangle ABC .



11. From the top of a tower the angle of depression of an object 300' from the foot of the tower is $19^\circ 52'$. What is the height of the tower?

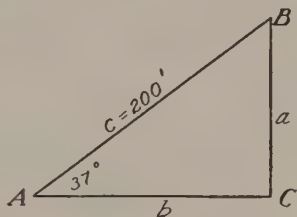
12. A navigator standing at water level in a passing boat finds that the angle of elevation of the top of a cliff known

to be 420' high is $11^{\circ} 7'$. A rocky reef extends 2000' out from the shore at the cliff; is the boat far enough away to clear it?

13. When the angle of elevation of the sun is $46^{\circ} 18'$, what is the length of the shadow cast by a pole 50' high?

THE SINE OF AN ANGLE

There are problems in finding sides and angles of a right triangle which cannot be solved by the use of the tangent ratio. Consider the following problem: In the right triangle ABC , $c = 200'$ and $A = 37^{\circ}$. Find a .



It is obvious that knowing the value of the ratio of a to b will not help us in finding a when we do not know b . We must have a ratio involving the known side c , such as the ratio $\frac{a}{c}$. This ratio of the side opposite the angle to the hypotenuse is called the sine of the angle. Sine A is usually abbreviated to $\sin A$.

$$\sin A = \frac{\text{opposite arm}}{\text{hypotenuse}} = \frac{a}{c}$$

The values of $\sin A$ for all values of A from 0° to 90° have been computed and arranged in a table which is found on pages 384-385. It is used in much the same way as the table of tangents.

EXERCISE 118

1. What is the value of $\sin A$ if $a = 45'$ and $c = 75'$?
2. Find $\sin B$ if $b = 6''$ and $c = 10''$.
3. If $a = 90$ and $b = 400$, find c and then find $\sin A$.

4. Find, from the table of sines, the value of $\sin A$ if A equals

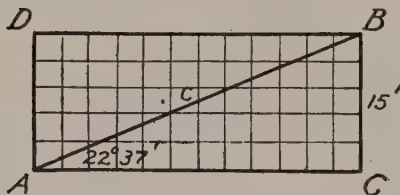
- | | | |
|--------------------|--------------------|--------------------|
| (a) 32° | (c) $77^\circ 30'$ | (e) $51^\circ 17'$ |
| (b) $44^\circ 10'$ | (d) $22^\circ 13'$ | (f) $84^\circ 9'$ |

5. Find, from the table of sines, the value of A in degrees and minutes if $\sin A$ equals

- | | | |
|------------|------------|------------|
| (a) 0.3173 | (c) 0.9124 | (e) 0.7042 |
| (b) 0.5495 | (d) 0.4423 | (f) 0.9350 |

6. In the right triangle ABC , $c = 200'$ and $A = 37^\circ$. Find a .

7. In the rectangle in the diagram find the length of c .



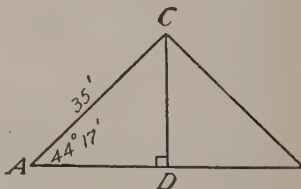
8. First find AC and then compute the length of c in Example 7, without making use of the sine. Compare the result with that obtained in Example 7.

9. A telephone pole is to be anchored by a wire from its top. If the pole is $32'$ high and the wire is to meet the ground at an angle of 52° , what length of wire will be required?

10. A ladder $25'$ long is placed with the top against a house and making an angle of $61^\circ 20'$ with the level ground. How far up the side of the house does it reach?

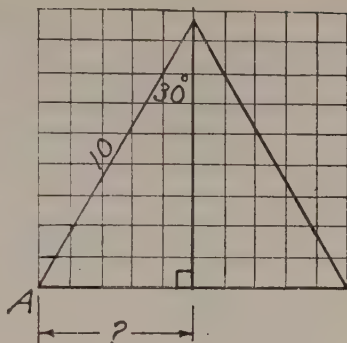
11. The angle of elevation of the top of a mountain, known to be $3335'$ high, from a point near its base is $32^\circ 15'$. What is the distance from the point of observation to the summit of the mountain in an air line?

12. ABC is an isosceles triangle and CD is its altitude. If AC is $35''$ and $A = 44^\circ 17'$, what is the length of CD ?

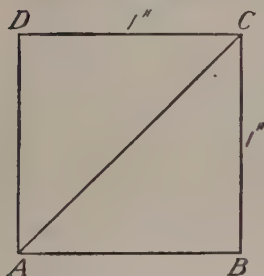


13. The altitude of an equilateral triangle forms a 30° angle with either side, and bisects the base. What is the sine of the 30° angle in the diagram?

14. Show that the altitude of the triangle in the above diagram is $5\sqrt{3}$, and find the tangent of the 30° angle.



15. The angle A of the above triangle is 60° . Find $\sin A$.



16. The diagonal of the square $ABCD$ forms an angle of 45° with the side. What is the tangent of the 45° angle in the diagram?

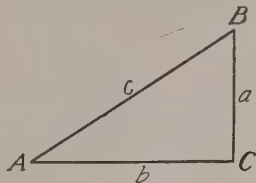
17. Show that $\sin 45^\circ = \frac{1}{2}\sqrt{2}$.

18. As a partial check on the preceding examples find from the tables the sine and tangent of 30° , of 60° , and of 45° .

THE COSINE OF AN ANGLE

In addition to the sine and the tangent there is a third ratio which is frequently used. This is the ratio of the side adjacent to the acute angle to the hypotenuse; it is called the cosine of the angle.

In the right triangle ABC , the ratio of b to c is the cosine of angle A ; that is, $\cos A = \frac{b}{c}$, or, as commonly abbreviated, $\cos A = \frac{b}{c}$.



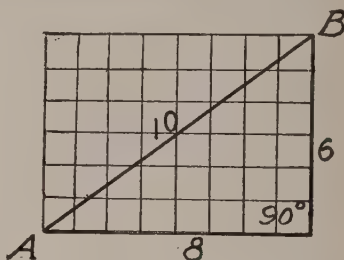
EXERCISE 119

1. Find the cosine of the angle A if $b = 14$ and $c = 20$.
 2. What is the value of $\cos A$ if $a = 56$ and $c = 140$?
 3. If $a = 6$ and $b = 8$, find the sine of angle A and the cosine of angle B ; find also $\cos A$ and $\sin B$.

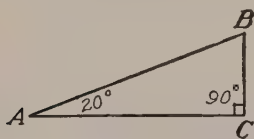
4. Fill each blank space with either the word sine or the word cosine:

$$\text{Sine } A = \frac{a}{c} = \text{---} B$$

$$\text{---} A = \frac{b}{c} = \text{---} B$$



If the angle A is 20° we know that the angle B is 70° , since the sum of the angles of any triangle is 180° .



The right angle is 90° so that the sum of the two acute angles of any right triangle is 180° less 90° , or 90° .

5. Find the missing angle A or B of a right triangle if

(a) $B = 30^\circ$

(d) $B = 40^\circ 30'$

(b) $A = 45^\circ$

(e) $A = 71^\circ 52'$

(c) $A = 12^\circ$

(f) $B = n^\circ$

Two angles whose sum is 90° are called *complements*, or *complementary angles*. The acute angles of any right triangle are complementary.

Because of the preceding facts a new table is not needed for cosines, since the cosine of any angle is the same as the sine of its complement. The cosine of 20° can be found in the table of sines in the same place as the sine of 70° ; the cosine of 70° can be found as the sine of 20° .

To use the table to find cosines one finds the angles in

degrees at the right, numbered up from the bottom of the page. Minutes are read from right to left, at the bottom of the page.

6. Find, from the table on page 384, the value of $\cos A$ when A equals

- | | |
|----------------|--------------------|
| (a) 15° | (d) $52^\circ 10'$ |
| (b) 30° | (e) $63^\circ 20'$ |
| (c) 45° | (f) $75^\circ 50'$ |

7. Find $\cos 37^\circ 17'$.

Solution: $\cos 37^\circ 10' = 0.7969$

$\cos 37^\circ 20' = 0.7951$

Difference = 0.0018

Note that the value of the cosine *decreases* as the angle *increases*.

To find $\cos 37^\circ 17'$ we subtract from the cosine of $37^\circ 10'$ $\frac{7}{10}$ of the difference 0.0018 , which is $.0013$

$$\begin{aligned}\therefore \cos 37^\circ 17' &= 0.7969 - 0.0013 \\ &= 0.7956\end{aligned}$$

8. Find the value of $\cos A$ if A equals

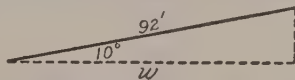
- (1) $28^\circ 15'$
- (2) $43^\circ 37'$
- (3) $56^\circ 28'$

9. In the right triangle ABC , $b = 210'$ and $A = 34^\circ 10'$. What is the length of c ?

10. If $c = 17.28$ and $B = 25^\circ 50'$, what is b ?

11. Mr. Collins' lot measures $92'$ wide along the surface of the ground, which slopes 10° .

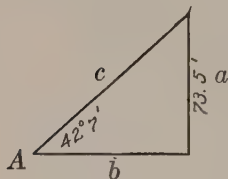
What is its width measured horizontally (the legal width)?



12. How long a wire cable is required to reach from the top of a wireless mast to a point on the ground $40'$ from the mast, if it makes an angle of $51^\circ 20'$ with the ground?

13. From the top of a hill the angle of depression of an object on the level plain from which the hill rises is $12^\circ 18'$. If the height of the hill is 1500', what is the distance in an air line from the top of the hill to the object?

EXERCISE 120



1. If in a right triangle, $a = 73.5'$ and $A = 42^\circ 7'$, find the length of b .

2. In the example above find, from the data given, the number of feet in c .

3. What is the angle of elevation of the sun when a tree 170' high casts a shadow of 150'?

4. What is the length of a diagonal of a rectangle which makes an angle of $23^\circ 15'$ with the base which is 18" long?

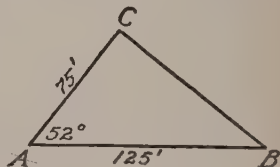
5. What is the altitude of the rectangle in Example 4?

6. A railroad roadbed rises 5' in every 100' horizontally. What is the angle of elevation of the roadbed?

7. An auto road over a hill makes an angle of $13^\circ 40'$ with the horizontal. When a car has gone one-half a mile up this hill, how much higher vertically is it than its starting point at the foot of the hill? Give answer in feet.

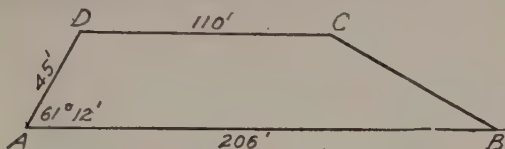
8. A three per cent grade is one that rises 3' vertically for each 100' horizontally. What is the angle of elevation of the grade?

9. ABC represents a triangular lot. What is the area of the lot? (From C let fall a perpendicular to AB ; this will be the altitude of the triangle.)

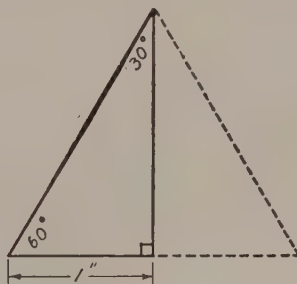
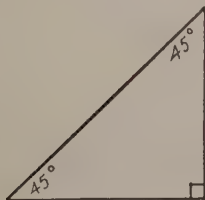


10. In the above diagram, let fall from B a perpendicular to AC . Find the length of this perpendicular and compute the area. Compare the answer with the answer to Example 9.

11. Find the area of the trapezoid in the diagram. (Hint: Drop a perpendicular from D to AB .)



12. What is the length of the hypotenuse of this 45° right triangle, arms each $10''$ long? Express $\sin 45^\circ$ and $\cos 45^\circ$ as radicals in simplest form.



13. What is the hypotenuse of this 30° , 60° right triangle (half of an equilateral triangle)? What is the value of $\cos 60^\circ$, as a radical?

CHAPTER XIV

SIMULTANEOUS EQUATIONS

EXPRESSING THE RELATIONSHIP OF NUMBERS

In Chapter I we expressed relations between numbers by means of (1) Rules and verbal directions, (2) Formulas and equations.

There are two other important ways of expressing relationship: (3) Tabulations or lists of corresponding values, (4) Graphs.

It is often very laborious to calculate an adequate tabulation of facts. Although it would take a very long time to obtain the values given in the table of square roots, etc. on page 378, the table includes only the whole numbers up to 50. A very incomplete tabulation, however, is often sufficient to enable us to draw a graph.

Example: Draw a graph, showing the cost of any number of articles at 5¢ each.

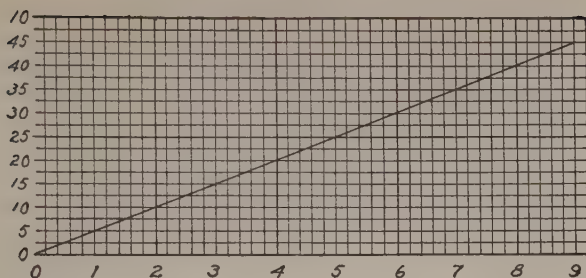
1. Rule: To find the total cost (C) multiply the cost (5) of a single article by the number of articles (n) purchased.

2. Formula: $C = 5n$

3. Tabulation of corresponding values:

C	0	5	10	15	20	25	30	35	40	45
n	0	1	2	3	4	5	6	7	8	9

This tabulation is obtained by giving n a succession of values in $C = 5n$, and finding the corresponding value of C ; thus, if $n = 3$, $C = 5n$, or $5(3)$ or 15.



4. The graph:

Horizontal Scale: Five spaces equal 1 unit.

Vertical Scale: Two spaces equal 5 cents.

EXERCISE 121

1. Draw a graph showing the distance traveled in any number of hours at the rate of 10 miles per hour:

(a) State the rule.

(b) Write the formula.

(c) Make the tabulation of values.

(d) Draw the graph, using the horizontal scale for the number of hours (t) and the vertical scale for the distance (d).

2. State the rule for finding the circumference of a circle when the diameter is known. What is the formula? Complete this tabulation of the circumferences of circles of given diameters, using $\pi = 3.14$ and calculating the values of C to the nearest tenth:

D	1	2	3	5	8	12	15
C	3.1	6.3					

3. Draw a graph of the formula $C = \pi D$ from $D = 1$ to $D = 10$.

4. Tabulate the interest on \$100 at 2% for periods of one year to ten years and draw the graph.

Example: Mrs. Ware bought a radio set on the installment plan, agreeing to pay \$5 down and \$2 a week.

(a) Write a formula for the total amount (A) which she will have paid after w weeks.

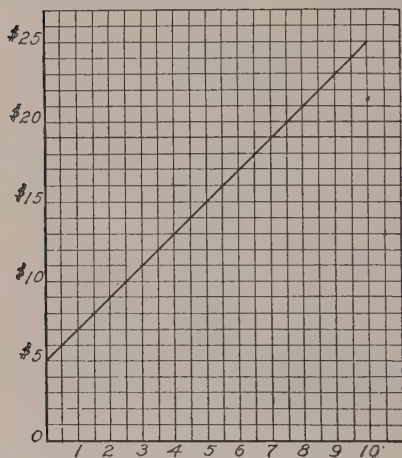
(b) Tabulate the values of A for each week up to 10 weeks.

(c) Draw a graph.

Solution: (a) Formula $A = 5 + 2w$

(b) Tabulation:

No. of Weeks	0	1	2	3	4	5	6	7	8	9	10
Amount Paid	5	7	9	11	13	15	17	19	21	23	25



(c) Graph (as shown).

5. A telegraph company charges 40 cents for a ten-word telegram and 5 cents for each additional word. What would a twelve-word telegram cost? a twenty-word telegram?

6. State a formula for the cost of a telegram (T) in terms of the number of words (w).

7. Tabulate the costs of telegrams at the above rates for any number of words from 10 to 20. Draw the graph.

8. A telephone company charges for a conversation between two cities 10 cents for the first 3 minutes or any fraction thereof, and 5 cents for each additional minute. Express this as a formula. Tabulate for any number of minutes from 3 to 10 and draw the graph.

Example: Draw a graph showing the relation between the readings of a Fahrenheit and Centigrade thermometer. (See the diagram and examples at the end of Chapter VIII.)

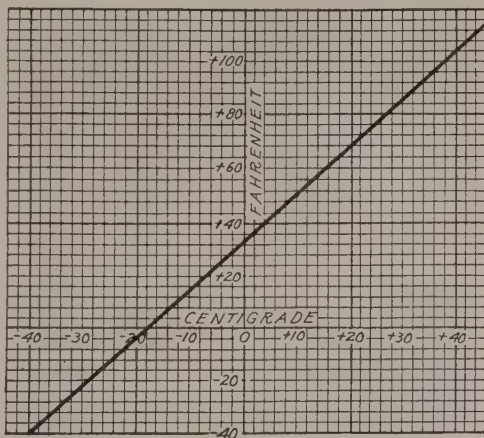
Formula: $F = \frac{9}{5}C + 32$

Tabulation:

F	- 22°	-	4°	14°	32°	50°	68°	86°	104°
C	- 30°	-	20°	- 10°	0°	10°	20°	30°	40°

Note that the reading on the thermometer scales may be either above zero (+) or below zero (-). We have found, in making the tabulation above, the values of F , corresponding to values of C from 30° below zero (-30°) to 40° above zero (+40°).

Graph: In the preceding graphs we have had only positive numbers to deal with; here we must make provision for negative numbers on both scales. We shall make use of the idea used in the number scale. If the scale is horizontal, numbers to the right of the zero point are positive, to the left are negative; if the scale is vertical, numbers above the zero point are positive, below the zero point are negative. The intersection of the two axes shall be the zero point for each.



Scale: Horizontal Scale: 1 space equals 2 degrees.

Vertical Scale: 1 space equals 4 degrees.

9. Answer the following questions from the graph above:

(1) What Fahrenheit reading corresponds to a reading of 4° above zero Centigrade?

(2) If the temperature is $+38^{\circ}$ Centigrade, what is the Fahrenheit reading?

(3) Fifteen degrees below zero Centigrade is the same as what temperature Fahrenheit?

(4) 59° Fahrenheit is the same as what Centigrade?

(5) Thirty-five degrees above zero Centigrade is the same as what temperature Fahrenheit?

10. The formula for obtaining the Centigrade reading, when the Fahrenheit is known, is $C = \frac{5}{9}(F - 32)$. Complete the following tabulation and then draw the graph:

C	-35°				-5°				
F	-31°	-13°	5°	23°	41°	59°	77°	104°	

Use the horizontal scale for the Fahrenheit readings and the vertical scale for the Centigrade.

Example: The difference of two numbers is 10. Write the formula or equation expressing the relationship of the two numbers and draw a graph.

Formula: $x - y = 10$

Tabulation:

x	-5	-2	0	5	10	15	20
y	-15	-12	-10	-5	0	5	10

Graph: Values of x on horizontal scale: 1 space = 1 unit

Values of y on vertical scale: 1 space = 1 unit

11. Answer the following questions from the graph on next page.

(1) What is the value of y when $x = 12$? Do these values satisfy the equation $x - y = 10$?

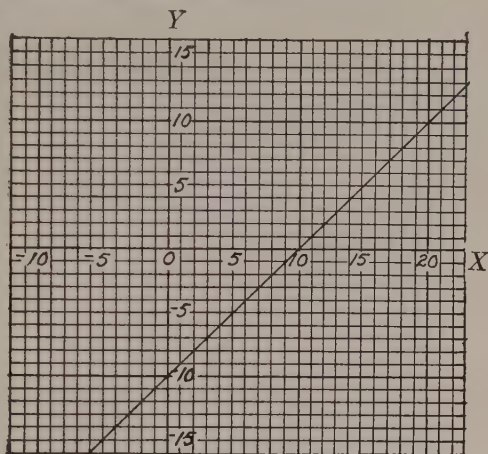
(2) If $x = -2$, what does y equal? Is the equation satisfied?

(3) If $y = 8$, what is the value of x ? Will the equation check with these values?

(4) At a certain point on the graph $x = 5$ and $y = 5$. Do these values of x and y satisfy the equation?

(5) If $x = -2$ and $y = -12$ the equation $x - y = 10$ is satisfied. Is there a point on the graph where x and y have those values?

From the above we learn two things: first, that any point on the graph gives a value for x and for y , which will satisfy the equation $x - y = 10$; and second, that any values of x and y which satisfy the equation $x - y = 10$ will be represented by a point on the graph.



12. The sum of two numbers (x, y) is 8.

(a) State this as an equation.

(b) Tabulate at least five sets of values for x and y .

(c) Draw a graph.

13. Draw the graph of the equation $x = 12 - 3y$.

You will have noticed that the graphs of the equations in the last four examples have been straight lines. This will always be the case with simple equations, like those of this exercise. This fact will help you in drawing the graph, for since it takes only two points to determine a straight line, it will be necessary in making your tabulation of values to find only two pairs of corresponding values. Fix the positions of the two points corresponding to these pairs of values and draw a straight line through them. It is advisable to find a

third pair of values as a check; the third point should be in a straight line with the other two.

14. Tabulate five or more sets of numbers such that one of the numbers (y) is twice the other (x). Could you extend the tabulation indefinitely? State an equation in terms of x and y , and draw a graph.

15. Draw the graph of $x + 3y = 11$, finding three points only.

16. Draw, on the same axes, the graphs of $x + y = 7$ and $2x = 3y - 1$.

Tabulations:

(1) $x + y = 7$	x	0	7	3
	y	7	0	4
(2) $2x = 3y - 1$	x	7	-8	$-\frac{1}{2}$
	y	5	-5	0

Graph: Scale on each axis 1 unit = 1 space

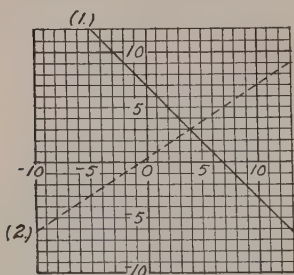
Full line is the graph of $x + y = 7$

Dotted line is the graph of $2x = 3y - 1$

Any point on Graph (1) gives values of x and y which satisfy the equation $x + y = 7$; any point on Graph (2) gives values of x and y which satisfy the equation $2x = 3y - 1$. Any point which is on both (1) and (2) will give values of x and y which satisfy both equations. The values of x and y given by the point of intersection in the diagram above are $x = 4$ and $y = 3$. Do these values satisfy both the given

equations? Two equations containing the same unknowns, each satisfied by the same values of the unknowns, are called *simultaneous equations*.

To solve such equations graphically, draw the graphs of



the two equations on the same axes; find the value of x and of y corresponding to the point of intersection. These values will be the roots of the two equations.

GRAPHS OF SIMULTANEOUS EQUATIONS

EXERCISE 122

Solve graphically and check your answers:

$$1. \begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$$

$$7. \begin{cases} 5x + 2y = 9 \\ 4y - x = 18 \end{cases}$$

$$2. \begin{cases} 2x + y = 13 \\ x + 2y = 14 \end{cases}$$

$$8. \begin{cases} x = \frac{1}{3}y + 3 \\ 7y = x - 18 \end{cases}$$

$$3. \begin{cases} 3x + 2y = 1 \\ 2x - y = 10 \end{cases}$$

$$9. \begin{cases} 4x = 3y \\ 2y = 3x - 1 \end{cases}$$

$$4. \begin{cases} 3x + 2y = 0 \\ x = 8 - 2y \end{cases}$$

$$10. \begin{cases} 6x + 2y = 14 \\ 2x - y = -1 \end{cases}$$

$$5. \begin{cases} x = 1 + 2y \\ y = .4x - 1 \end{cases}$$

$$6. \begin{cases} 11 - y = 3x \\ x + 3y - 13 = 0 \end{cases}$$

$$11. \begin{cases} x + y = -5.5 \\ x - y = 0.9 \end{cases}$$

ELIMINATION BY ADDITION OR SUBTRACTION

Although, as we have seen, simultaneous equations in two unknowns can be solved graphically, this method of solution has disadvantages. The results, even when the work is carefully done, depend upon measurement and are only approximate. We have a quicker and more accurate method of solution in which we use only the four fundamental operations of algebra. This method, which is called *elimination by addition or subtraction*, is illustrated in the following examples:

Example 1: Solve for x and y and check:

$$\begin{cases} 2x + 3y = 97 \\ 5x + 3y = 148 \end{cases}$$

Solution:

$$(1) \quad 2x + 3y = 97$$

$$(2) \quad 5x + 3y = 148$$

Subtracting left and right members,

$$-3x = -51$$

and we have eliminated y . What axiom was used?

$$\therefore x = 17$$

Substituting this value of x in Equation 1,

$$(1) \quad 34 + 3y = 97$$

$$3y = 63$$

$$y = 21$$

$$\begin{cases} x = 17 \\ y = 21 \end{cases} \quad \text{Ans.}$$

Check:

Left Member	Right	Left Member	Right
(1) $2x + 3y$	97	(2) $5x + 3y$	148
$34 + 63$	97	$85 + 63$	148
$97 \longleftrightarrow 97$		$148 \longleftrightarrow 148$	

This example illustrates elimination by subtraction.

Example 2: Solve for r and s and check:

$$\begin{cases} 3r - 4s = 9 \\ 2r + 4s = 86 \end{cases}$$

Solution: (1) $3r - 4s = 9$

(2) $2r + 4s = 86$

Adding, we have

$$5r = 95$$

and we have eliminated the s . State the axiom used.

$$\therefore r = 19$$

Substituting this value of r in Equation 2, we have

$$(2) \quad 38 + 4s = 86$$

$$\begin{cases} 4s = 48 \\ s = 12 \end{cases} \quad \begin{cases} r = 19 \\ s = 12 \end{cases} \quad \text{Ans.}$$

Check:

Left	Right	Left	Right
(1) $3r - 4s$	9	(2) $2r + 4s$	86
$57 - 48$	9	$38 + 48$	86
$9 \longleftarrow \longrightarrow$	9	$86 \longleftarrow \longrightarrow$	86

This example illustrates elimination by addition.

Example 3: Solve for a and b and check:

$$\begin{cases} 5a + 7b = 12 \\ 2a + 21b = 23 \end{cases}$$

With these two equations we cannot eliminate an unknown quantity either by adding or by subtracting the left members. If, however, we multiply each member of the first equation by 3, then the coefficients of the b terms will be the same and we can eliminate the b terms by subtraction.

Solution:

$$(1) \quad 5a + 7b = 12$$

$$(2) \quad 2a + 21b = 23$$

Multiplying each member of Equation 1 by 3,

$$(3) \quad 15a + 21b = 36$$

$$(2) \quad 2a + 21b = 23$$

Subtracting,

$$13a = 13$$

$$\therefore a = 1$$

Substituting this value of a in Equation 2,

$$2 + 21b = 23$$

$$21b = 21 \quad \begin{cases} a = 1 \\ b = 1 \end{cases}$$

$$b = 1 \quad \text{Ans.}$$

Check:

Left	Right	Left	Right
(1) $5a + 7b$	12	(2) $2a + 21b$	23
$5 + 7$	12	$2 + 21$	23
$12 \longleftarrow \longrightarrow$	12	$23 \longleftarrow \longrightarrow$	23

Example 4: Solve for m and n and check:

$$\begin{cases} 5m + 7n = 31 \\ 7m - 5n = -1 \end{cases}$$

Solution:

$$(1) \quad 5m + 7n = 31$$

$$(2) \quad 7m - 5n = -1$$

Multiplying each member of Equation 1 by 7, and each member of Equation 2 by 5, in order to make the coefficients of the m terms the same, we have

$$(3) \quad 35m + 49n = 217$$

$$(4) \quad 35m - 25n = -5$$

Subtracting,

$$74n = 222$$

$$n = 3$$

Substituting this value of n in equation (1),

$$5m + 21 = 31$$

$$5m = 10 \quad \begin{cases} m = 2 \end{cases}$$

$$m = 2 \quad \begin{cases} n = 3 \end{cases} \quad \text{Ans.}$$

Check:

Left	Right	Left	Right
(1) $5m + 7n$	31	(2) $7m - 5n$	-1
$10 + 21$	31	$14 - 15$	-1
$31 \longleftrightarrow 31$		$-1 \longleftrightarrow -1$	

Oral Exercise

Solve by addition or subtraction:

1. $\begin{cases} x + y = 12 \\ x - y = 2 \end{cases}$

4. $\begin{cases} 2x + 3y = 11 \\ 2x - 3y = 5 \end{cases}$

2. $\begin{cases} a + b = 8 \\ a - b = 4 \end{cases}$

5. $\begin{cases} 2c + d = 7 \\ 2c - d = 3 \end{cases}$

3. $\begin{cases} m + 2n = 11 \\ m - 2n = -1 \end{cases}$

6. $\begin{cases} 2x + y = 800 \\ x + y = 500 \end{cases}$

$$7. \begin{cases} b + 3a = 13 \\ b + 2a = 9 \end{cases}$$

$$9. \begin{cases} 2m + 3n = 7000 \\ m + 3n = 4000 \end{cases}$$

$$8. \begin{cases} y + 7z = 9 \\ y - z = 1 \end{cases}$$

$$10. \begin{cases} 2r - s = 11 \\ 2r + 5s = 29 \end{cases}$$

In each of the following examples, multiply both members of either or of both of the equations by such numbers that the coefficients of y , in each example, will be numerically the same:

$$11. \begin{cases} 2x + 3y = 8 \\ x + y = 2 \end{cases}$$

$$15. \begin{cases} 10y = 11 - 5x \\ 2y = 7 - 4x \end{cases}$$

$$12. \begin{cases} 3x + y = 7 \\ 5x + 4y = 10 \end{cases}$$

$$16. \begin{cases} 3x + 4y - 15 = 0 \\ 2x + 3y - 10 = 0 \end{cases}$$

$$13. \begin{cases} 2x + 7y = 13 \\ 3x - y = 7 \end{cases}$$

$$17. \begin{cases} 3x + 5y = 13 \\ 4x - 2y + 5 = 0 \end{cases}$$

$$14. \begin{cases} 7x - 4y = 8 \\ 2x + 2y = 7 \end{cases}$$

$$18. \begin{cases} 11x + 14y = 29 \\ 7x - 16y = -7 \end{cases}$$

Additional practice may be had by making the coefficients of x , in the above examples, numerically the same.

EXERCISE 123

Solve the following equations by addition or subtraction and check your answers:

$$1. \begin{cases} 3a + 7b = 23 \\ 3a - 2b = 5 \end{cases}$$

$$6. \begin{cases} 4a + 5b = -33 \\ 4a - b = -3 \end{cases}$$

$$2. \begin{cases} 2r + 3s = 19 \\ 2r - s = 7 \end{cases}$$

$$7. \begin{cases} 2x - 7s = 3 \\ 5x - 7s = -3 \end{cases}$$

$$3. \begin{cases} 5m + n = 15 \\ 5m - 4n = -10 \end{cases}$$

$$8. \begin{cases} x + 5y = 110 \\ 2x - 17y = 166 \end{cases}$$

$$4. \begin{cases} 5x + y = 11 \\ 3x - 2y = 4 \end{cases}$$

$$9. \begin{cases} m - 9n = 4 \\ 3m + n = 348 \end{cases}$$

$$5. \begin{cases} 3m + 2n = 22 \\ 3m - n = 25 \end{cases}$$

$$10. \begin{cases} 10d + c = 3 \\ 4d + 6c = 4 \end{cases}$$

$$11. \begin{cases} 2p - 4q = -6 \\ p + 2q = 405 \end{cases}$$

$$12. \begin{cases} 5y - 6z = 1 \\ 2y + z = -20 \end{cases}$$

$$13. \begin{cases} 2x + 3y = 12 \\ 3x + 2y = 13 \end{cases}$$

$$14. \begin{cases} 2r - 3s = 2 \\ 3r - 2s = 8 \end{cases}$$

$$15. \begin{cases} 3v + 8t = 64 \\ v - 2t = 12 \end{cases}$$

$$16. \begin{cases} 15m + 11n = 4 \\ 5m + 3n = 2 \end{cases}$$

$$17. \begin{cases} 3x + 7y = 5 \\ 7x - 3y = 2 \end{cases}$$

$$18. \begin{cases} 8b_0 + 8b_2 = 144 \\ 2b_0 = 3 + b_2 \end{cases}$$

$$19. \begin{cases} 11x - 12y = 19 \\ 17x - 9y = -47 \end{cases}$$

$$20. \begin{cases} 3w + 2w' + 18 = 0 \\ 8 + 3w' = 5w \end{cases}$$

$$21. \begin{cases} 3x - 2y = 4875 \\ 5y - 3x = 1875 \end{cases}$$

$$22. \begin{cases} y = 7x - 3 \\ 2y - 5x - 12 = 0 \end{cases}$$

$$23. \begin{cases} 10t + u = 10u + t - 18 \\ 3t + 4u = 29 \end{cases}$$

$$24. \begin{cases} 8x - 3y = 10 \\ 2x + 9y = 35 \end{cases}$$

$$25. \begin{cases} 10s = 7l + 2 \\ 10l = 9s + 13 \end{cases}$$

$$26. \begin{cases} 20 - 7w = 20 - 5l \\ 12 + 3w = 13 + 2l \end{cases}$$

$$27. \begin{cases} \frac{r}{2} + \frac{s}{3} = 7 \\ \frac{r}{3} + \frac{s}{4} = 5 \end{cases}$$

$$27. \begin{cases} \frac{r}{3} + \frac{s}{4} = 5 \end{cases}$$

$$28. \begin{cases} 1.2x - 1.3y = 1 \\ 2.5x - 2.6y = 2.3 \end{cases}$$

$$29. \begin{cases} 0.7y - 1.2x = -0.47 \\ 0.3y + 0.4x = 0.53 \end{cases}$$

$$30. \begin{cases} 2a = \frac{2b}{3} + 4 \\ 3a = \frac{3b}{4} + 12 \end{cases}$$

31. The sum of two numbers (x, y) is 72 and the difference of the numbers is 6. Write two equations and find the numbers.

ELIMINATION BY SUBSTITUTION

A second method of solving a pair of simultaneous equations in two unknowns is called elimination by substitution. We find in one of the given equations the value of one unknown in terms of the other unknown and substitute this value in the other equation.

Example 1: Solve for x and y and check:

$$\begin{cases} x = 7y \\ 3x - y = 60 \end{cases}$$

Solution:

$$(1) \quad x = 7y$$

$$(2) \quad 3x - y = 60$$

Equation (1) gives the value of x in terms of y , $x = 7y$.
Substituting $7y$ in place of x in the second equation,

$$(3) \quad 3(7y) - y = 60$$

and we have eliminated the x .

$$21y - y = 60$$

$$20y = 60$$

$$y = 3$$

Substituting this value of y in equation (1),

$$\begin{aligned} x &= 7(3) \\ x &= 21 \end{aligned} \quad \begin{cases} x = 21 \\ y = 3 \end{cases} \text{ Ans.}$$

Check:

	Left	Right		Left	Right
(1)	x	$7y$	(2)	$3x - y$	60
	21	$7(3)$		$63 - 3$	60
	$21 \longleftrightarrow 21$			$60 \longleftrightarrow 60$	

Example 2: Solve for x and y and check:

$$\begin{cases} x + 5y = 110 \\ 2x - 17y = 166 \end{cases}$$

Solution:

$$(1) \quad x + 5y = 110$$

$$(2) \quad 2x - 17y = 166$$

Finding from equation (1) the value of x in terms of y ,

$$(3) \quad x = 110 - 5y$$

Substituting this value of x for x in equation (2),

$$\begin{aligned} 2(110 - 5y) - 17y &= 166 \\ 220 - 10y - 17y &= 166 \\ -10y - 17y &= 166 - 220 \\ -27y &= -54 \\ y &= 2 \end{aligned}$$

Substituting this value of y for y in equation (3),

$$\begin{aligned} x &= 110 - 10 \\ x &= 100 \end{aligned} \quad \begin{cases} x = 100 \\ y = 2 \end{cases} \quad \text{Ans.}$$

Check:

	Left	Right		Left	Right
(1)	$x + 5y$	110	(2)	$2x - 17y$	166
	$100 + 10$	110		$200 - 34$	166
	$110 \longleftrightarrow 110$			$166 \longleftrightarrow 166$	

Oral Exercise

Solve the following equations, using the method of substitution:

- | | |
|--|--|
| <p>1. $\begin{cases} y = 3x \\ x + y = 8 \end{cases}$</p> <p>2. $\begin{cases} a = 2b \\ 3b + a = 10 \end{cases}$</p> <p>3. $\begin{cases} m = 3n \\ 5n - m = 16 \end{cases}$</p> <p>4. $\begin{cases} r = 4s \\ 5s - r = 1 \end{cases}$</p> <p>5. $\begin{cases} x = \frac{1}{2}y \\ 2x + 2y = 20 \end{cases}$</p> <p>6. $\begin{cases} v = 16t \\ 3v + 2t = 100 \end{cases}$</p> | <p>7. $\begin{cases} y = 7x \\ 3x + 4y = 31 \end{cases}$</p> <p>8. $\begin{cases} y = 3x + 1 \\ x + y = 9 \end{cases}$</p> <p>9. $\begin{cases} m = n + 2 \\ 5n + m = 14 \end{cases}$</p> <p>10. $\begin{cases} r = s - 1 \\ 5s + r = 11 \end{cases}$</p> <p>11. $\begin{cases} x = 2z + 3 \\ x + 4z = 21 \end{cases}$</p> <p>12. $\begin{cases} x = y + 2 \\ 2x + y = 13 \end{cases}$</p> |
|--|--|

In each of the following equations find the value of x in terms of y :

13. $x + 2y = 7$

17. $2x + 3y = 9$

14. $x - 3y = 8$

18. $3x - 2y = 6$

15. $x + 7y = 10$

19. $3x + y = -8$

16. $x - 5y = -8$

20. $3x - 5 = 7y$

If additional work is desired, find the value of y in terms of x in the preceding examples.

EXERCISE 124

Solve the following equations, using the method of substitution and check your answers:

1. $\begin{cases} y = 3x \\ x + 3y = 10 \end{cases}$

6. $\begin{cases} r = 2s - 30 \\ 6s - 5r = -50 \end{cases}$

2. $\begin{cases} x = 3y \\ 5x - 8y = 14 \end{cases}$

7. $\begin{cases} m + n = 100 \\ 2m + 3n = 900 \end{cases}$

3. $\begin{cases} y = 3x + 2 \\ 3x + 5y = 28 \end{cases}$

8. $\begin{cases} 2x + y = 25 \\ 3x = 2y + 6 \end{cases}$

4. $\begin{cases} x = 2y - 3 \\ 7x - 10y = 3 \end{cases}$

5. $\begin{cases} a = -2b \\ 3a + 8b = 5 \end{cases}$

9. $\begin{cases} 6x - 25y = -1 \\ 3x - 5y = 7 \end{cases}$

10. $\begin{cases} y = 3x \\ 2(x - y) = 3\left(2x - 3y + \frac{17}{3}\right) \end{cases}$

11. Solve by the substitution method, and check by solving by the addition or subtraction method:

$$\begin{cases} 3l = 4w + 4 \\ 2l + 2w = 26 \end{cases}$$

12. Solve by both methods, as in Example 11:

$$\begin{cases} 2x + 3y = 11 \\ y = \frac{1}{2}x + 6 \end{cases}$$

13. Solve Example 12 by drawing the graphs of the two equations.

14. The length (l) of a rectangle is 1" more than double the width (w) of the rectangle; and the perimeter of the rectangle is 44". Find l and w .

EXERCISE 125

Solve the following equations by either method of elimination and check the answers:

$$1. \begin{cases} 10t + u = 10u + t + 45 \\ t + u = 9 \end{cases}$$

$$2. \begin{cases} y = x \\ \frac{2y - 3x + 8}{3} = 1 - \frac{3y - x - 12}{3} \end{cases}$$

$$3. \begin{cases} 3r + \frac{1}{3}t - 2 = 0 \\ \frac{1}{2}r + 2t - \frac{37}{6} = 0 \end{cases}$$

$$4. \begin{cases} \frac{x+2}{y+2} = \frac{1}{2} \\ \frac{x+7}{y+7} = \frac{2}{3} \end{cases}$$

$$6. \begin{cases} x - y = 9 \\ 3 + \frac{5}{11}y = \frac{3x}{7} \end{cases}$$

$$5. \begin{cases} \frac{x+1}{y} = \frac{1}{3} \\ \frac{x}{y-1} = \frac{1}{4} \end{cases}$$

$$7. \begin{cases} x + y = 1\frac{2}{3} \\ x = 3\left(y + \frac{1}{3}\right) \end{cases}$$

$$8. \begin{cases} \frac{3a-2b}{4} = \frac{2a-3b}{2} + 3 \\ \frac{a}{3} + \frac{b}{3} = \frac{13}{3} \end{cases}$$

$$9. \begin{cases} (x+3)(y-1) = xy + 5 \\ (x-2)(y+2) = xy \end{cases}$$

$$10. \begin{cases} x = 5050 - y \\ .04x + .05y = 220 \end{cases}$$

$$11. \begin{cases} x = 12 - y \\ \frac{10x + y}{x - y} = 21 \end{cases}$$

$$12. \begin{cases} y = 5x + 3 \\ y - 3(5x - 17) = -x - (y - 13) + 36 \end{cases}$$

$$13. \begin{cases} b_1 = 3b_2 - 2 \\ \frac{5}{2}(b_1 + b_2) = 35 \end{cases}$$

$$14. \begin{cases} \frac{1}{2}(A + B) - \frac{1}{3}(A - B) = 8 \\ \frac{1}{2}(A - B) + \frac{1}{3}(A + B) = 14 \end{cases}$$

$$15. \begin{cases} m : n + 1 = 4 : 7 \\ m - 1 : n = 5 : 9 \end{cases}$$

$$16. \begin{cases} y = 3x + 1 \\ \frac{1}{2}(2x + 3y) - \frac{1}{3}(3x - 2y) = 21\frac{2}{3} \end{cases}$$

$$17. \begin{cases} \frac{a - 2}{3} + \frac{b - 4}{5} = 3 \\ \frac{a + 4}{4} - \frac{2b - 4}{7} = 1 \end{cases}$$

$$18. \begin{cases} \frac{2x + 3y}{7} - \frac{x - 3y - 8}{5} = \frac{137}{35} \\ \frac{3x - y}{4} + \frac{8 - 2x}{5} = \frac{43}{20} \end{cases}$$

$$19. \begin{cases} \frac{2m}{3} + \frac{3n}{4} = 48 \\ \frac{5m}{12} - \frac{7n}{8} = -13 \end{cases}$$

$$20. \begin{cases} \frac{5y}{x - 2} + \frac{4x - 7}{2x - 4} = \frac{21}{6x - 12} \\ \frac{5x}{y - 2} + \frac{4y - 7}{2y - 4} = \frac{21}{6y - 12} \end{cases}$$

$$21. \begin{cases} 2x + \frac{y}{3} = 4004 \\ 2y + \frac{x}{3} = 3864 \end{cases}$$

$$22. \begin{cases} \frac{x+2y}{3} + \frac{4x+5y}{6} = \frac{7}{20} \\ x - 2y = 0 \end{cases}$$

$$23. \begin{cases} \frac{1}{2}x = y \\ \frac{1}{2}y = x + y + \frac{5}{6} \end{cases}$$

$$24. \begin{cases} \frac{2A+B-7}{6-3A+5B} = \frac{1}{7} \\ \frac{3A-4B+7}{16-A-B} = \frac{8}{11} \end{cases}$$

$$25. \begin{cases} (x+2)(x-3) - (y-6)(y+7) = x^2 - y^2 \\ (x-6)(x+7) + (y+2)(y-3) = x^2 + y^2 \end{cases}$$

$$26. \begin{cases} 5(2x+3y) - 7(3x+4y) = -36 \\ 7(2x-3y) + 5(3x-4y) = 396 \end{cases}$$

$$27. \begin{cases} \frac{y}{x} = 0.333 \\ \frac{y}{1000-x} = 0.111 \end{cases}$$

$$28. \begin{cases} \frac{2a}{3} + \frac{3b-1}{7} - 6 = 0 \\ \frac{2b}{3} + \frac{3a-1}{7} - \frac{121}{21} = 0 \end{cases}$$

$$29. \begin{cases} 0.3x = 0.7y - 0.42 \\ 0.7x = 0.3y + 0.62 \end{cases}$$

$$30. \begin{cases} r + s = .0015 \\ r = 3(s + 0.0001) \end{cases}$$

$$31. \begin{cases} 3(x-2y) = -5x - 3y + 10 \\ x - (3y - 2x) + 7 = x - 6y - 6(y-7) \end{cases}$$

$$32. \begin{cases} \frac{3t+7}{4v-8} = \frac{-9}{10} \\ v : t = 3 : -4 \end{cases} \quad 33. \begin{cases} 6x + 5y = 62 \\ \frac{1}{2}x = \frac{1}{3}y + \frac{31}{60} \end{cases}$$

$$34. \begin{cases} y - \frac{1}{2}x = x - \frac{1}{2}y + 12 \\ x : y = -1 : 6 \end{cases}$$

$$35. \begin{cases} (x + 1.5)(y - 1.5) = xy - 5.25 \\ (x - 1.5)(y + 2.5) = xy + 7.75 \end{cases}$$

$$36. \begin{cases} \frac{3A}{5} + \frac{5B}{3} = 2 \\ 15(A - B) = 9A + (B + \frac{2}{5}) \end{cases}$$

Hint: Use the method of substitution at once, before removing fractions.

$$37. \begin{cases} \frac{576}{s} = t \\ \frac{576}{s - 24} = 2t \end{cases}$$

$$38. \begin{cases} \frac{y}{x} = 1 \\ \frac{y}{1000 - x} = 0.577 \end{cases}$$

$$39. \begin{cases} x = \frac{2y + 7}{5} \\ y = \frac{8x + 7}{5} \end{cases}$$

$$40. \begin{cases} \pi D + \pi D' = 132 \\ D - D' = 28 \end{cases} \quad \text{Let } \pi = 3\frac{1}{7}$$

$$41. \begin{cases} 2\pi r_1 = 2\pi r_2 + 12.5664 \\ r_2 = 22 - r_1 \end{cases} \quad \text{Let } \pi = 3.1416$$

Sometimes we find fractional equations where the two unknowns appear only in the denominators, which are usually monomials. With equations of this kind it is better not to clear of fractions. The way to handle such equations is illustrated in the following example:

Example 1: Solve for m and n and check:

$$\begin{cases} \frac{2}{m} + \frac{3}{n} = 13 \\ \frac{5}{m} + \frac{1}{n} = 13 \end{cases}$$

Solution:

$$(1) \quad \frac{2}{m} + \frac{3}{n} = 13$$

$$(2) \quad \frac{5}{m} + \frac{1}{n} = 13$$

Multiplying the members of Equation (2) by 3,

$$(3) \quad \frac{15}{m} + \frac{3}{n} = 39$$

$$(1) \quad \frac{2}{m} + \frac{3}{n} = 13$$

Subtracting member from member,

$$\frac{13}{m} = 26$$

$$\therefore 13 = 26m$$

$$\frac{1}{2} = m$$

Substituting this value of m in equation (1),

$$\frac{2}{\frac{1}{2}} + \frac{3}{n} = 13$$

$$4 + \frac{3}{n} = 13$$

$$\frac{3}{n} = 9$$

$$\begin{aligned} 3 &= 9n \\ \frac{1}{3} &= n \end{aligned} \quad \left\{ \begin{aligned} m &= \frac{1}{2} \\ n &= \frac{1}{3} \end{aligned} \right. \quad \text{Ans.}$$

Check:

Left		Right
(1) $\frac{2}{m} + \frac{3}{n}$		13
$\frac{2}{\frac{1}{2}} + \frac{3}{\frac{1}{3}}$		13
$4 + 9$		13
13	←→	13

Left		Right
(2) $\frac{5}{m} + \frac{1}{n}$		13
$\frac{5}{\frac{1}{2}} + \frac{1}{\frac{1}{3}}$		13
$10 + 3$		13
13	←→	13

EXERCISE 126

Solve the following equations and check your answers:

$$1. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{3}{4} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{4} \end{cases}$$

$$2. \begin{cases} \frac{1}{a} + \frac{1}{b} = 5 \\ \frac{1}{a} - \frac{1}{b} = 1 \end{cases}$$

$$3. \begin{cases} \frac{3}{r} + \frac{1}{s} = 1\frac{5}{8} \\ \frac{2}{r} + \frac{1}{s} = 1\frac{1}{3} \end{cases}$$

$$4. \begin{cases} \frac{5}{x} + \frac{2}{y} = 7 \\ \frac{3}{x} + \frac{2}{y} = 5 \end{cases}$$

$$5. \begin{cases} \frac{2}{m} + \frac{2}{n} = 2 \\ \frac{3}{m} + \frac{1}{n} = 2 \end{cases}$$

$$6. \begin{cases} \frac{48}{m} + \frac{33}{n} = 5 \\ \frac{24}{m} + \frac{11}{n} = 2 \end{cases}$$

$$7. \begin{cases} \frac{12}{x} - \frac{3}{y} = 14 \\ \frac{15}{x} + \frac{6}{y} = 30\frac{1}{2} \end{cases}$$

$$8. \begin{cases} \frac{3}{p} + \frac{4}{q} = 7 \\ \frac{1}{p} + \frac{6}{q} = \frac{35}{6} \end{cases}$$

$$*9. \begin{cases} \frac{5}{3r} + \frac{1}{s} = \frac{8}{3} \\ \frac{7}{2r} + \frac{1}{2s} = 4 \end{cases}$$

Hint: In above example multiply each side of the second equation by 2.

$$10. \begin{cases} \frac{4}{3x} + \frac{1}{y} = \frac{5}{6} \\ \frac{3}{2x} - \frac{1}{2y} = \frac{1}{8} \end{cases}$$

$$11. \begin{cases} \frac{1}{x} = 7 + \frac{2}{3y} \\ \frac{1}{y} = 7 - \frac{4}{9x} \end{cases}$$

$$*12. \begin{cases} \frac{2}{3a} + \frac{3}{2b} = 4 \\ \frac{2}{3b} + \frac{3}{2a} = \frac{43}{12} \end{cases}$$

In the next two examples find the values of x and y to two decimal places:

$$13. \begin{cases} \frac{3}{x} + \frac{1}{y} = 0.727 \\ \frac{2}{x} + \frac{1}{2y} = 0.652 \end{cases}$$

$$14. \begin{cases} \frac{3}{x} + \frac{2}{y} = 3.276 \\ \frac{2}{x} + \frac{3}{y} = 3.146 \end{cases}$$

EXERCISE 127

Solve the following equations for x and y and check your answers:

$$1. \begin{cases} ax + by = a + b \\ x - y = 0 \end{cases}$$

$$4. \begin{cases} bx - ay = ab \\ x : y = 3a : 2b \end{cases}$$

$$2. \begin{cases} x : y = 4 : 1 \\ 5x + 4y = 24a \end{cases}$$

$$5. \begin{cases} mx + ny = 2a \\ 2mx - 3ny = -a \end{cases}$$

$$3. \begin{cases} 2bx + 3ay = 13ab \\ 3bx + 2ay = 12ab \end{cases}$$

$$6. \begin{cases} ax - by = a^2 + b^2 \\ x + y = 2a \end{cases}$$

$$7. \begin{cases} x = a + 3y \\ ax - a^2 = 3(a + b)y - 3b^2 \end{cases}$$

$$8. \begin{cases} \frac{x}{m} + \frac{y}{n} = m + n \\ \frac{x + y}{mn} = 2 \end{cases}$$

$$9. \begin{cases} \frac{m}{x} + \frac{n}{y} = \frac{m + n}{mn} \\ \frac{m}{x} - \frac{n}{y} = \frac{m - n}{mn} \end{cases}$$

$$10. \begin{cases} x = a + b + y \\ 2c(x - a) = 3(y - b) \end{cases}$$

$$11. \begin{cases} (a + b)x - (a - b)y = 4ab \\ x + y = 2a \end{cases}$$

12. Solve for a and l in terms of the other letters:

$$\begin{cases} s = \frac{n}{2}(a + l) \\ l = a + (n - 1)d \end{cases}$$

13. Solve for n_1 and n_2 :

$$\begin{cases} n_1 + n_2 = s \\ n_1 - n_2 = d \end{cases}$$

14. Solve for l and w :

$$\begin{cases} 2l + 2w = p \\ (l - 1)(w + 1) = lw + k \end{cases}$$

15. Solve for b_1 and b_2 :

$$\begin{cases} \frac{1}{2} h(b_1 + b_2) = A \\ \frac{1}{2} h(b_1 - b_2) = \frac{1}{2} A \end{cases}$$

16. Solve for x and y :

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Oral Exercise

Express the following statements as equations, using two unknown quantities:

1. The sum of two numbers is 10.
2. The difference of two numbers is 5.
3. The larger number is $7\frac{1}{2}$ more than the smaller number.
4. One number exceeds the other by 8.
5. One number is 5 less than the other.
6. The denominator of a certain fraction exceeds the numerator by 6.
7. The numerator of a certain fraction is $\frac{3}{4}$ of the denominator.
8. The sum of the ages of A and B is 85 years.
9. A is 17 years older than B.
10. The perimeter of a certain rectangle is 49.
11. The length of a certain rectangle exceeds twice the width by 3.
12. If the larger of two numbers is divided by the smaller, the quotient is 3.
13. Two-thirds of the larger of two numbers exceeds three-quarters of the smaller number by 7.
14. The value of the contents of a purse containing dimes and nickels is \$1.30.
15. In 5 years A will be twice as old as B.
16. If the larger of two numbers is divided by the smaller, the quotient is 2 and the remainder is 2.
17. The area of a rectangular field is 84 sq. rd. (l, w).
18. If n represents the numerator, and d the denominator of a fraction, what is the fraction?

19. What does the fraction $\frac{n}{d}$ become if one is added to its numerator and two to its denominator?

20. How much does a grocer get for x pounds of tea at 35 cents a pound, and y pounds of coffee at 29 cents a pound?

21. Express as an equation: If the length of a rectangle is diminished by 4 inches and the width is increased by 5 inches, the area remains unchanged.

22. How many miles does a train travel in t hours at the rate of r miles an hour?

23. What does the fraction $\frac{n}{d}$ become, if a is subtracted from its numerator and b is added to its denominator?

24. Express as an equation: If each of the terms of a fraction is increased by 3, the value of the fraction becomes $\frac{1}{2}$.

25. A man makes two investments, one producing 4% interest and the other 5%. His total income is \$200. Express the above as an equation.

EXERCISE 128

1. The sum of two numbers is 99 and their difference is 45. What are the numbers?

2. Separate 120 into two parts so that if the larger part is divided by the smaller, the quotient will be 7.

3. The length of a rectangle is twice the width. The perimeter of the rectangle is 42". What are the dimensions of the rectangle?

4. A man found a purse containing 15 coins, — quarters and dimes; the value of the coins was \$2.25. How many coins of each kind were there?

5. I am thinking of two numbers whose sum is 52. One of the numbers is $3\frac{1}{3}$ times as large as the other. What are the numbers?

6. The sum of the numerator and the denominator of a certain fraction is 34. If 6 be subtracted from the numerator and 7 be added to the denominator, the value of the new fraction is $\frac{1}{4}$. Find the fraction.

7. I purchased 24 postage stamps for 93 cents. Some of the stamps cost 5 cents each, the rest 2 cents each. How many of each kind did I buy?

8. A grocery store sells two grades of coffee, — one at 29 cents a pound, the other at 35 cents a pound. On a certain day the store sold 113 pounds of coffee for which it received \$35.23. How many pounds of each kind of coffee did the store sell?

9. A and B together have \$12,100. A invests his money at 6% and B his at 5%. They have the same income. How many dollars does each invest?

10. Separate 1728 into two parts so that the sum of $\frac{1}{4}$ of the larger part and $\frac{1}{8}$ of the smaller part shall be 341.

11. The difference in the radii of two circles is 7". The sum of their two circumferences is 220". Find the radius of each circle. (Take $\pi = 3\frac{1}{7}$.)

12. The sum of the radii of two circles is 25 yd. The difference in their circumferences is 15.708 yd. Find the radius of each circle. (Take $\pi = 3.1416$.)

13. The larger of two numbers is 8 more than the smaller. One-third of the larger number added to one-fourth of the smaller is 19. Find the two numbers.

14. A ticket seller sold \$685.00 worth of tickets, some at 25 cents each and the rest at 15 cents each. How many tickets of each kind did he sell, if the total number sold was 2900?

15. At an entertainment which was attended both by adults and by children, the receipts from the sale of tickets were \$500. The cost of a ticket for an adult was 50 cents, and for a child 25 cents. How many of the 1200 who purchased tickets were children?

16. If the numerator and the denominator of a certain fraction are each increased by 1, the value of the resulting fraction is $\frac{1}{2}$. If the numerator is diminished by 12 and the denominator by 13, the value of the resulting fraction is $\frac{1}{6}$. What is the fraction?

17. If the numerator of a certain fraction is increased by 1, the value of the fraction is $\frac{2}{3}$. If the denominator of the fraction is increased by 7, the value of the fraction is $\frac{1}{2}$. What is the fraction?

18. The sum of the length and the width of a rectangle is 33'. If the rectangle were 6" shorter and 3" wider its area would be the same. Find the length and the width of the rectangle.

19. After making a purchase I received 85 cents in change. There were 12 coins in all, — nickels and dimes. How many coins of each kind did I receive?

20. Three-fifths of A's age now is the same as B's age nine years ago. The sum of their present ages is 81 years. How old is each now?

21. A grocer has two grades of coffee, — one worth 23 cents a pound, the other 29. How many pounds of each kind shall he use to get a mixture of 90 pounds worth 25 cents a pound?

22. The sum of two numbers is a and their difference is b . What are the numbers?

Oral Exercise

1. What is the number whose tens' digit is 7 and units' digit 3?

2. What would this number become if the order of the digits were reversed?

3. In Roman numerals what is meant by X? by V? by XV? What sign is understood between X and V?

4. The number 98 may be written as a sum, $90 + 8$. In a similar way write 52 as a sum. Write 753 as a sum of three terms.

5. Although our modern numerals make use of the addition principle, as did the Roman, their distinctive feature is place value. What does the 7 actually represent in the number 47? in 74? in 1709? Write a number in which 7 will represent seven thousand.

6. When the Romans wrote the symbols for "five" and "one" alongside each other, they meant *five plus one* or *six*. Compare the meanings of (a) V and 5; (b) VI and 51; (c) VII and 511.

7. What is the value of a number whose tens' digit is t and units' digit u ?

8. What would be the value of this number if the digits were interchanged?

9. State as an equation: If the two digits of a certain number are interchanged, the value of the number is increased by 36.

10. State as an equation: If a certain number is divided by the sum of its two digits, the quotient is 5.

11. An escalator, or moving stairway, carries people at the rate of 3 miles per hour. A person walks ahead on the escalator at the rate of 2 miles per hour. What is his actual rate of progress?

12. A college crew can row in still water at the rate of 10 miles per hour. How fast can they row downstream if the current is flowing at the rate of 2 miles per hour? How fast can they row upstream?

13. How long would it take the crew to row 4 miles downstream? 4 miles upstream?

14. A motor boat making r miles per hour would take how long to go $\frac{r}{2}$ miles? 1 mile?

15. How long would it take the motor boat to go a distance of d miles downstream in a river flowing c miles per hour? d miles upstream?

16. Express as an equation: It takes the motor boat just as long to go 2 miles upstream as 5 miles downstream.

17. If you can mow your lawn in 5 hours, what part of the lawn can you mow in 1 hour? in 3 hours?

18. James can do a certain piece of work in 3 days and Bill in 4 days. What part of the work can each do in 1 day? What part of the work can both do in 1 day?

19. Jack can do a piece of work in m days and Tom can do it in n days. State as an equation: Jack and Tom can by working together do the work in 6 days.

20. A ferry boat can fill its water tanks from one line of 3" hose in 10 minutes. How long would it take two lines of 3" hose to fill the tanks? three lines of 3" hose?

21. If a certain pipe can fill a tank in m minutes, what fraction of the tank can it fill in 1 minute?

22. If another pipe can fill this tank in n minutes, what fraction of the tank can be filled in 1 minute by the two pipes together?

EXERCISE 129

1. The sum of the digits of a two-digit number is 13. If 27 be subtracted from the number, the order of the digits will be reversed. What is the number?

2. The units' digit of a certain number is 2 less than the tens' digit. If the order of the digits is reversed the sum of the new number and the original number is 88. Find the number.

3. A man can row 12 miles downstream in $1\frac{1}{2}$ hours, and 12 miles upstream in 4 hours. What is his rate of rowing in still water, and what is the rate of the current?

4. A steamer goes 60 miles downstream in 3 hours and makes the return trip in 5 hours. What is the rate of the current, and how many miles per hour can the steamer travel in still water?

5. A man made a trip of 9 miles in $1\frac{1}{2}$ hours. He walked part of the way at a rate of 3 miles per hour, and rode the rest of the way at 9 miles per hour. How many hours did he walk?

6. If the speed of a train were 2 miles per hour more, it would take $\frac{1}{3}$ of an hour less time to travel a certain distance; if the speed were 4 miles per hour less, it would take $\frac{7}{9}$ of an hour more to travel the same distance. What is the speed of the train and the time required?

7. If a certain number whose tens' digit is 2 greater than its units' digit is divided by the sum of its digits, the quotient is $6\frac{1}{4}$. What is the number?

8. What is that number whose tens' digit exceeds twice the units' digit by 2, and which is 5 less than eight times the sum of its digits?

9. A cistern can be filled by two pipes, running together, in 18 minutes. If one pipe runs for 10 minutes and the other for 15 minutes the cistern will be $\frac{2}{3}$ filled. How many minutes will each pipe alone take to fill the cistern?

10. A and B together can do a piece of work in 2 days. They work together 1 day, then A quits work and it takes B 3 days longer to finish the work. How many days would each require alone to do the work?

11. Two automobiles leave a certain city at the same time, going in the same direction. They are 21 miles apart in 7 hours. At what rate is each traveling if the speed of one is $\frac{8}{9}$ of the speed of the other?

12. Two trains leave the same place at the same time; one going north, the other south. In six hours they are 498 miles apart. How many miles per hour is each train traveling if the difference in their speeds is 9 miles per hour?

13. The sum of the hundreds' digit and the units' digit of a three-digit number is 7. The tens' digit is 7. If 297 be added to the number, the order of the digits will be reversed. What is the number?

14. A tank is filled by two pipes. If the first pipe runs for 8 hours and then the second for 12 hours the tank will be full; if the first pipe runs for 12 hours, then it will require only 6 hours for the second pipe alone to fill the tank. How many hours are required for each pipe to fill the tank alone?

15. It took a man 2 hours to row 4 miles up a stream, but he rowed back in 30 minutes. What is the rate of the current?

EXERCISE 130

1. The larger of two numbers is $2\frac{1}{3}$ times the smaller. Five-sixths of the smaller number is 11 less than $\frac{3}{4}$ of the larger number. Find the numbers.

2. A collection of 12 coins was worth \$2.25. It consisted of half dollars, dimes, and 3 nickels. How many half dollars and how many dimes were there?

3. The numerator of a fraction is $\frac{3}{4}$ of the denominator. If 6 be subtracted from the numerator and 9 be added to three times the denominator, the value of the fraction is $\frac{1}{15}$. What is the fraction?

4. A rectangular flower bed is surrounded by a path 2' wide. The area of the path is 112 square feet. If the length of the flower bed is 4 feet more than its width, what are the dimensions of the bed?

5. The sum of the radii of two circles is 15". The difference in their circumferences is 18.8496". Find the two radii. [Take $\pi = 3.1416$.]

6. A invested a certain sum at 4% and B a different amount at 5%. Their total annual income is \$122. If A had received 5% on his investment and B 4% on his, their total annual income would have been \$121. How much did each invest?

7. Of 665 pupils who took algebra in a certain school, 497 passed at the end of the term. If 80% of the boys passed

and 70% of the girls, how many boys and how many girls took algebra?

8. If the larger of two numbers is divided by the smaller, the quotient is 2 and the remainder 270. If 3 times the smaller number is divided by the larger, the quotient is 1 and the remainder 459. What are the numbers?

9. If the sum of two numbers is divided by the larger number the quotient is 1 and the remainder 5236; if the difference of the two numbers is divided by the smaller number, the quotient is 3. What are the numbers?

10. A man paid a debt of \$114 with five-dollar bills and two-dollar bills. The number of five-dollar bills was 6 more than the number of two-dollar bills. How many bills of each denomination were used?

11. A and B working together can do a certain piece of work in 6 days. If A works alone for 4 days and B then works alone for 3 days, they will have done three-fifths of the work. In how many days can each do the work alone?

12. Two boys together can paint a boat in 4 days. After they have worked together for 3 days one boy stops work and the other has to work 3 days longer to finish the job. How many days would each boy require for the painting of the boat, if he did it alone?

13. The lower base of a trapezoid is 9' longer than the upper base; the altitude of the trapezoid is 10'. If the area is 225 square feet, what are the bases of the trapezoid?

14. The larger of two numbers exceeds the smaller number by $\frac{1}{6}$. One-third of the larger number is equal to one-half of the smaller number. What are the numbers?

15. A farmer hired 6 men and 4 boys for one day for \$34. Later he hired 5 men and 8 boys, at the same rates, for \$36.50 a day. How much does a man receive per day?

16. The denominator of a certain fraction is 13 more than twice the numerator. If the numerator and the denominator

are each increased by 1 the value of the fraction is then $\frac{2}{5}$. Find the fraction.

17. The bases of two rectangles are 52' and 42' respectively. The area of the first rectangle is 344 square feet less than the area of the second rectangle and its perimeter is 4' less than the perimeter of the second rectangle. Find the altitude of the two rectangles.

18. There are three rectangles which have the same area. The second is 6" shorter and 4" wider than the first; the third is 3" longer and $1\frac{1}{3}$ " narrower than the first. Find the length and the width of each rectangle.

19. A farmer purchased two cows for \$255. He sold one cow at a profit of 20% but lost 8% on the other. He made \$16 by the two transactions. How much did he pay for each of the cows?

20. A dealer in candy has on hand chocolates which he sells at 49 cents a pound and bonbons which he sells at 59 cents a pound. He wishes to make a mixture of the two (75 pounds in all) to sell at 55 cents a pound. How many pounds of each kind of candy shall he use?

21. Two boys start from two points 100 yards apart and run toward each other; they meet in 6 seconds. If one boy runs for 4 seconds only and the other for 6 seconds only, they would still be 20 yards apart. How long does it take each to run 100 yards?

22. A certain number consists of two digits, of which the tens' digit is the larger. If the number is divided by the sum of its digits, the quotient is 9 and the remainder 1; if the number formed by reversing the order of the digits is divided by the difference of the digits, the quotient is 2 and the remainder 3. What is the number?

23. If $\frac{1}{3}$ of the sum of two numbers be added to $\frac{1}{5}$ of the difference of the two numbers, the result will exceed the smaller number by $18\frac{1}{3}$; if $\frac{1}{5}$ of the sum of the two numbers

be added to $\frac{1}{3}$ of the difference of the two numbers, the result will exceed the smaller number by $11\frac{2}{3}$. What are the numbers?

24. If the length of a certain rectangle were 3" less and its width 4" more, it would then be a square whose area would be 28 square inches more than the area of the rectangle. What are the dimensions of the rectangle?

25. A part of \$18,700 is invested at $3\frac{1}{2}\%$ and the rest at 4%. If the total annual income from the two investments is \$703, how much is invested at each per cent?

26. If the larger of two numbers is diminished by 6 and the smaller is increased by 4, the product of the resulting numbers is the same as the product of the original numbers; but if the larger number is diminished by 4 and the smaller is increased by 6 the product of the resulting numbers is 52 more than the product of the original numbers. Find the two numbers.

27. A trapezoid has an area of 168 square inches. Its altitude is 12 inches and its lower base exceeds its upper base by 10 inches. Find the lengths of the two bases.

28. A is now 45 years old and B 25. A's age x years ago was the same as B's will be y years hence; B's age x years ago was $\frac{1}{3}$ of what A's will be y years hence. Find x and y .

29. Two triangles whose altitudes are respectively 12" and 20" have the same area. The base of the first triangle is 6.4" longer than the base of the second triangle. Find the bases of the two triangles.

30. A baker has 27 pounds of a mixture of two kinds of fancy cakes. One kind he usually sells at 75 cents a pound, the other at 90 cents a pound. If he sells the mixture at 80 cents a pound without losing any money, how many pounds of each kind of cakes did he put in the mixture?

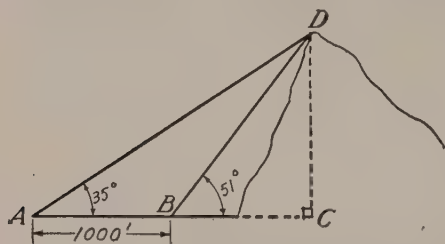
31. Two pumps, working together, can empty a tank in 12 hours. After 8 hours' pumping, one pump stops and the

other continues pumping until the tank is empty, which requires 14 hours more. How long will it take each pump by itself to empty the tank?

PROBLEMS INVOLVING TRIGONOMETRY

Sometimes when we are solving problems in finding heights and distances by the use of the trigonometric ratios we have to solve equations in which there are two unknown quantities. The following examples will show how these problems are solved:

1. At a certain point the angle of elevation of the top of a hill was 35° . At another point, 1000 feet nearer the hill,



the angle of elevation was 51° . What is the height of the hill?

In the diagram A is the point at which the angle of elevation is 35° ; B is 1000 feet nearer the hill, and the angle of elevation at B is 51° . DC , which represents the height of the hill, is at right angles with the line AB . Call DC x , and BC y .

Now in the triangle BCD , $\frac{x}{y} = \tan 51^\circ$, and in the triangle ACD , $\frac{x}{1000 + y} = \tan 35^\circ$.

We find from the tables on page 382, that $\tan 35^\circ = 0.700$ and $\tan 51^\circ = 1.235$.

Substituting these two values in the two equations,

$$\frac{x}{y} = 1.235$$

$$\frac{x}{1000 + y} = 0.700$$

Solving these equations:

$$x = 1.235 y$$

$$x = 700 + 0.700 y$$

$$\therefore 1.235 y = 700 + 0.700 y$$

$$0.535 y = 700$$

$$y = 1308 \text{ (correct to 4 significant figures)}$$

$$x = 700 + 0.700 (1308)$$

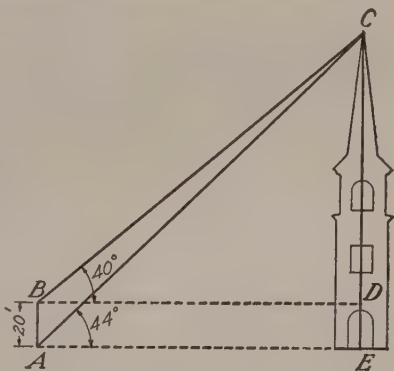
$$= 700 + 916$$

$$= 1616$$

Therefore the height of the hill is 1616 feet.

2. From the lower window of a house, the angle of elevation of the top of a church tower is 44° ; and from a window 20 feet above the first, the angle of elevation of the top of the church tower is 40° . How far is the church from the house?

In the diagram, let A be the point at which the angle of elevation of C , the top of the church steeple, is 44° . Let B , twenty feet above A , be the point at which the angle of elevation of C is 40° . We wish to find BD (or AE) which represents the distance of the house from the church. Call CD x and call BD and AE each y .



In the triangle AEC

$$\frac{x + 20}{y} = \tan 44^\circ$$

In the triangle CDB

$$\frac{x}{y} = \tan 40^\circ$$

Substituting for $\tan 40^\circ$ and $\tan 44^\circ$ their values as found in the tables, we have

$$\frac{x + 20}{y} = 0.966$$

$$\frac{x}{y} = 0.839$$

$$x + 20 = 0.966 y$$

$$x = 0.839 y$$

$$\therefore 0.839 y + 20 = 0.966 y$$

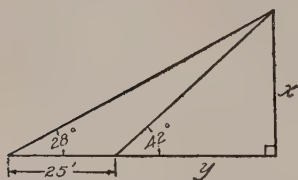
$$0.127 y = 20$$

$$y = 157, \text{ the number of feet from the house to the church.}$$

EXERCISE 131

1. The angle of elevation of the top of a church tower is 51° from a point on the ground and is 39° from a point 42 feet directly above the first point. How high is the church tower? Make a scale drawing and read your answer from the drawing.

2. Work out Example 1, making use of the trigonometric ratios.



3. Find in the annexed diagram the lengths of x and y .

4. From the top of a tower 150 feet high, the angles of depression of the top and the bottom of another tower standing on the same horizontal plane are $23\frac{1}{2}^\circ$ and $34\frac{1}{2}^\circ$ respectively. What is the height of the second tower?

5. Draw to scale a triangle ABC . AB is 18 inches, the angle A is 35° , the angle B is 47° . With compasses let fall a perpendicular CD from C to AB . What is the length of CD ?

6. In the diagram for Example 5, call CD , h ; AD , x , and DB , $18 - x$. Using the table of trigonometric functions find x and h .

7. Construct to scale, as accurately as possible, with protractor and ruler, a triangle ABC ; the line AB is $10''$ in length, angle A is 30° and the angle B is 135° . With compasses drop a perpendicular CD from C to the line AB produced. Measure CD .

8. Using the diagram for Example 7 and a table of the values of the trigonometric functions, compute the length of CD . Compare answer with answer to Example 7.

GENERAL REVIEW

Factoring and Fractions.

1. What are the three prime factors of 30? of 78? of 125?
 2. What is the highest common factor of all the terms of $9a^4 + 6a^3b + 15a^2b^2$?
 3. Factor $9a^2x + 27ax^2 + 27x^3$.
 4. Factor $m^2 - 25$.
 5. Factor $3r^2 + r - 2$.
 6. Factor $12 - 27n^2$.
 7. Factor $8y^2 - 24y + 18$.
 8. Reduce to lowest terms $\frac{(x+a)^2}{x^2 - a^2}$. Check when $a = 2$, $x = 3$.
 9. Reduce to lowest terms $\frac{1 - n^2}{n - n^2}$. Check when $n = 2$.
 10. Reduce to lowest terms $\frac{a^2 + 2ab + b^2 - c^2}{a + b - c}$. Check when $a = 1$, $b = 2$, $c = 1$.
 11. Simplify and find the value of $1 + \frac{1}{1 + \frac{1}{4}}$.
- Perform the operations indicated, and check if possible:
12. $\frac{3a - 2}{4} + 5$.

$$13. \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

$$14. \frac{2p^2 + q^2}{p^2 - q^2} + \frac{q}{p + q}.$$

$$15. \frac{12n^2 + 16n - 3}{4n^2 - 9} \cdot \frac{2n^2 + 5n - 12}{30n^2 + n - 1} \div \frac{n^2 + 4n}{5n^2 + n}.$$

$$16. \frac{a^2x - ax^2}{a^2 - x^2} \cdot \frac{a^2 - ax - 2x^2}{2a^2 - ax - x^2} \cdot \frac{4a^2 - x^2}{2a^3 - 5a^2x + 2ax^2}.$$

$$17. 2\frac{3}{8} + \frac{1}{4} - .04\frac{3}{4} + \frac{7}{16} - 2.56\frac{1}{4}.$$

Radicals

18. By inspection find the value of $2\sqrt{25}$, $\frac{1}{a}\sqrt{4a^2}$, and $\sqrt[3]{\frac{27}{n^6}}$.

19. By inspection find the value of $\sqrt[3]{.008} + \sqrt{.09} + \sqrt{1}$.

20. From the tables of square roots find the value of $\sqrt{h}$ if $h = 13.8$.

21. Simplify each radical: $\sqrt{18}$, $\sqrt{28}$, $\sqrt{2n^2}$, $\sqrt{\frac{3s^2}{4}}$.

22. Express each radical in simplest form:

$$\sqrt[3]{56}, \sqrt[3]{8r}, \sqrt{\frac{7}{16}}.$$

23. Simplify each radical, and collect like terms:

$$\sqrt{27} + \sqrt{300} - \sqrt{\frac{1}{3}} + \sqrt{5\frac{1}{3}}$$

24. If the arms of a right triangle are r and $2r$, express the hypotenuse as a radical in simplest form.

25. If the base of an isosceles triangle is $2m$ and each of the other sides is $3m$, what is the altitude?

26. Rationalize the denominator of each fraction:

$$\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{m}}.$$

27. Rationalize the denominator of $\frac{1}{2\sqrt{5}}$ and find the value of the fraction if $\sqrt{5} = 2.236$.

28. If $x = 3 + \sqrt{7}$, find whether or not $x^2 + 2$ is equal to $6x$.

29. Show that $x = -4 - \sqrt{3}$ satisfies the equation $x^2 + 8x + 13 = 0$.

30. Simplify, and obtain the value of each result to three significant figures, if $\sqrt{6} = 2.449$:

$$\sqrt{600}, \sqrt{\frac{1}{8}}, \frac{\sqrt{3}}{\sqrt{2}}, \sqrt{\frac{2}{3}}.$$

31. Find the square root of 31.84 to four significant figures,

(a) by the square root process,

(b) by the use of the tables.

Proportion and Trigonometry

32. Arrange the four numbers 5, 16, 4, and 20 so as to form a proportion, beginning with 20.

33. Using the same four numbers write seven other proportions.

34. Solve for x , and check, $x : 14 - 2x = 2 : 3$.

35. Draw a circular graph to show how the money spent by a town was divided:

Police and Fire Dept.	\$35,000
Parks	10,000
Roads and Sewers	25,000
Schools	85,000
Interest on Debt	22,000
Miscellaneous	23,000

36. In enlarging a photograph $3'' \times 5''$ the width is to be $10''$. What will be the length?

37. On squared paper draw a right triangle with base $AC = 15$ units and altitude $CB = 8$ units. What is the

tangent of the angle A ? Express it as a decimal, and find A in degrees and minutes from the tables.

38. If $b = 15$ and $a = 8$, find the hypotenuse c . What is the sine of A ? Find A from the tables, and check with the previous example.

39. Referring to Example 37, fill in the blank spaces:

$$\cos ? = \frac{15}{17}, \quad \tan B = \frac{?}{?}, \quad \sin B = \frac{?}{?}$$

40. If in a right triangle $c = 120'$ and $A = 34^\circ$, find a and also find b . Make a drawing of the triangle on a scale of $1''$ to $20'$, and measure the sides as a rough check.

41. Check the preceding example more accurately by substituting in the formula $c^2 = a^2 + b^2$.

Equations and Formulas

42. Solve and check: $\frac{n+1}{2} - \frac{n-1}{3} = 1.$

43. Solve for n : $l = a + (n-1)d.$

44. Solve for x : $\frac{2x-k}{x-k} - \frac{x-k}{x+k} = 1.$

45. Solve for x and y , and check:
$$\begin{cases} \frac{x+1}{3} - \frac{y}{4} + 1 = 0 \\ \frac{y}{3} = \frac{3x+1}{4} \end{cases}$$

46. Solve for x and y , and check:
$$\begin{cases} x = \frac{2}{3}y - 4 \\ y = 0.5x + 2. \end{cases}$$

47. Draw the graphs of the equations in the preceding example.

48. Solve for a and b , and check:
$$\begin{cases} \frac{1}{a} + \frac{4}{b} = 1 \\ \frac{2}{a} + \frac{2}{b} = 1 \end{cases}$$

*49. Given $\frac{x}{a} + \frac{y}{b} = 1$. Find the values of a and b if when x is 0, $y = 2$ and when y is 0, $x = 4$.

*50. Given $\frac{x}{a} + \frac{y}{b} = 1$. Find the values of a and b if the graph of the equation passes through the points $(x = 5, y = 0)$ and $(x = 0, y = 1)$.

*51. Find x and y if the three expressions $2x + y$, $4 - y$, and $3x + 5y + 1$ all have the same value.

52. The gear of a bicycle can be found by means of the formula $G = \frac{fD}{r}$. Find the value of G when $f = 26$, $D = 28$, and $r = 8$. (f and r represent the number of teeth in the front and rear sprockets respectively, and D is the wheel diameter.)

53. Change the subject of the formula $G = \frac{fD}{r}$ to f , and use the new formula to find f if $D = 28$, $r = 10$, and $G = 70$.

*54. If you own a bicycle, count the teeth in the two sprockets and calculate the gear.

55. Try to solve for x and y the equations:

$$\begin{cases} 2y - x = 1 \\ 5 + x = 2y \end{cases}$$

56. Draw the graph of each of the equations in Example 55. (Since these equations cannot be satisfied by any pair of values of x and y they are not simultaneous, and are said to be inconsistent. One of the equations in fact contradicts the other.)

57. Try to solve for x and y the equations

$$\begin{cases} y = \frac{1}{3}x + 4 \\ x - 3y = -12 \end{cases}$$

58. Draw the graph of each of the equations in Example 57. (The second equation can be derived from the first,

and therefore merely repeats the first equation. The two equations are not independent of each other, and are said to be dependent equations.)

Draw the graphs of the equations in Examples 59–62, and determine for each example whether the two equations are simultaneous, inconsistent, or dependent:

$$59. \begin{cases} x + y = 10 \\ y = 6 - x \end{cases}$$

$$60. \begin{cases} x + y = 4 \\ x + 4 = y \end{cases}$$

$$61. \begin{cases} 2y - 6x + 4 = 0 \\ 3x = y + 2 \end{cases}$$

$$62. \begin{cases} x - y = 4 \\ x + 4 = y \end{cases}$$

63. The rule for parcel post rates to the third zone is: "The postage is 6 cents for the first pound and 2 cents for each additional pound." What would be the postage on a package weighing 2 pounds? 3 pounds? Make a tabulation showing the postage on packages weighing 1, 2, 3, 4, 5, 6, 7, and 8 pounds.

64. Draw a graph based on your tabulation in 63.

65. Write a formula for the postage (p) to the third zone on a parcel weighing n pounds.

66. Parcel post rates to the first and second zones may be tabulated as follows:

Weight	1 lb.	2 lb.	3 lb.	4 lb.	5 lb.	6 lb.	7 lb.	etc.
Postage	\$.05	.06	.07	.08	.09	.10	.11	

(a) State the rule on which this tabulation is based.

(b) Make a formula for the first and second zones.

(c) Draw a graph based on the tabulation.

67. An automobile starts out with 12 gallons of gasoline and uses $\frac{1}{20}$ of a gallon for each mile. How many gallons will

be left after 100 miles? Write a formula for the number of gallons left after m miles.

68. The Commonwealth Water Co. charges (quarterly) $22\frac{1}{2}$ cents per 100 cubic feet of water used, plus a fixed service charge of \$1.25. What will Mr. Drake pay if he uses 1200 cubic feet? Write a formula giving the amount charged (c) to a consumer who uses n hundreds of cubic feet.

69. An electric light company charges \$1 (monthly) for using 10 kilowatts or less, plus 10 cents for each additional kilowatt. Express this as a formula.

70. Write a formula for the total surface (s) of a cube in terms of its edge (e). Change the subject of the formula to e .

71. What is the edge of a cube if its total surface is 6 square inches? 24 square inches? 150 sq. centimeters?

72. Write a formula for the volume of a cube (V , e), and change the subject of the formula to e .

73. What is the edge of a cube if its volume is 27 cubic feet? 1000 cubic centimeters? 0.008 cubic inch?

*74. When a is very small as compared to 1, the expression $1 + na$ may be used as a close approximation for $(1 + a)^n$. Find the values of $1 + na$ and $(1 + a)^n$ when $a = .01$ and $n = 2$. By how much do the values differ?

*75. What would be the error in taking $1 + na$ as the value of $(1 + a)^n$ if $a = .02$ and $n = 3$?

76. Divide $n^3 + 27$ by $n + 3$. Check by substituting $n = 2$.

77. Multiply $a - b + c$ by $a - b - c$.

78. Express the ratio of 40 to 62.5 as a decimal.

79. In a chemist's mixture alcohol and water are in the ratio 95 : 5. After adding 200 ounces of water they will be in the ratio 45 : 55. How many ounces of each are there now?

80. How much water must be added to one gallon of alcohol 95% pure in order to reduce it to a 50% solution?

81. Express as a radical in simplest form the altitude and area of an equilateral triangle, side 12".

82. Simplify each term and add:

$$\sqrt{\frac{4}{3}} + \sqrt{\frac{3}{4}} + \frac{10}{\sqrt{3}} - \sqrt{\frac{1}{3}}$$

83. Find two values of x that satisfy the equation

$$(4 + x)(4 - x) = 7. \text{ Check.}$$

84. Using the square root tables solve the equation $x^2 = 19.15$.

85. Two weights (x , y) balance when one is 12" and the other 8" from the fulcrum. If the second weight is increased by 5 pounds and placed 2" nearer to the fulcrum it still balances the first weight. Find the two weights.

86. The flag pole in Independence Park casts a shadow 48' 6" long at the same time that a man 6' tall casts a shadow 2' 10" long. What is the height of the pole?

87. Factor the left member and solve for h :

$$2lh + 2wh = S.$$

88. Solve for f : $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$.

89. Bell metal is an alloy containing 16 parts of copper and 5 parts of tin. How many pounds of each are in a church-bell weighing 1.4 tons?

90. A bin is to be constructed to hold 9 tons of coal. If the bin is to be 5 feet deep and twice as long as it is wide, what will be its length and width? (Allow 40 cubic feet to the ton.)

91. One arm of a right triangle is 30", and the hypotenuse is 18" longer than the other arm. Find the sides of the triangle, and its area.

AN EXAMINATION

*Time, two hours.*PART I (*Answer all*)

1. Solve and check: $x : 14 - x = 4 : 3$

2. Solve and check: $2 = \frac{3}{x} + 5$

3. Express the ratio of an inch to a yard as a decimal (three decimal places), and as a per cent.

4. State in words the meaning of the formula $c^2 = a^2 + b^2$ and find the diagonal of a rectangle which is 30" x 16".

5. With the aid of tables find the angle formed by the diagonal and the longer side of the above rectangle.

6. Solve and check: $\frac{x+3}{x+1} - \frac{x-2}{x-1} = \frac{2x+3}{x^2-1}$

7. Solve for r :

$$p = \frac{A}{1+rt}$$

8. Find the value of r in the preceding question if $p = \$300$, $A = \$360$, and $t = 4$.

9. Solve for x and y , and check:
$$\begin{cases} x + 3y = 9 \\ \frac{1}{2}y = x - 2 \end{cases}$$

10. Draw the graphs of the equations in the preceding question.

PART II (*Answer any four*)1. (a) How tall is a flag-pole that casts a shadow 84 ft. long, if at the same time a post $4\frac{1}{2}$ feet high casts a shadow 6 feet long?

(b) The side of two squares measure 75' and 50'. What is the ratio of their perimeters? of their areas?

2. Find a fraction such that if 4 is added to the numerator the value of the fraction becomes $\frac{3}{4}$. If 3 is added to the denominator the fraction is equal to one-third.

3. (a) Simplify $\sqrt{\frac{3s^2}{4}}$ and $\sqrt{\frac{d^2}{2}}$

(b) Rationalize the denominator and find the value to two decimal places:

$$\frac{\sqrt{5}}{\sqrt{2}}$$

4. In a right triangle having its hypotenuse 13" and one arm 12", calculate the other arm. Draw the triangle to scale on cross-section paper. Find the sine, cosine, and tangent of the smallest angle of the triangle, expressing each to the nearest thousandth.

5. (With the aid of tables) The top of the Statue of Liberty is 301 feet above the surface of the water; from a small boat the angle of elevation of the top of the statue is measured and found to be 12° . How far is the boat from the foot of the statue?

NOTE. Extra credit will be allowed for checking No. 5 by finding the hypotenuse of the triangle and then applying $c^2 = a^2 + b^2$.

CHAPTER XV

QUADRATIC EQUATIONS

SOLUTION BY COMPLETING THE SQUARE

After reviewing the solution of equations based upon the study of square root the pupil may try the following examples, all of them to be solved by taking the square root of each member of the equation. Each example has two solutions. Check them separately.

EXERCISE 132

1. $x^2 = 25$

5. $(5y - 2)^2 = 49$

2. $9z^2 = 4$

6. $x^2 + 2x + 1 = 9$

3. $\frac{1}{4}D^2 = 81$

7. $y^2 - 8y + 16 = 1$

4. $(x - 3)^2 = 16$

8. $(x + 3)^2 = (2x - 5)^2$

Ans. $x = -1$ or $x = 7$

Ans. $x = 8$ or $x = \frac{2}{3}$

These are called *quadratic equations* because the square of the unknown quantity appears in them. If no terms in the first power of x appear, the equation is of the type $ax^2 + b = c$ and may be solved as in Chapter XII. If there is a term in x as well as a term in x^2 the equation is said to be a complete quadratic equation. In general the left member will not be a square, but may be made into a square by adding a properly chosen number to both members. (See Chap. XII.)

9. What number could be added to each of the following to complete a trinomial square:

$$x^2 + 4x + ? \quad x^2 - 12x + ? \quad x^2 + 20x + ?$$

10. Complete a square in each equation:

$$x^2 + 2x + () = 15 + () \quad x^2 + 6x + = 40 + ()$$

Rule for completing the square when the coefficient of x^2 is 1: *Add the square of half the coefficient of x to each member of the equation.*

Example 1: Solve and check:

$$x^2 + 14x - 15 = 0$$

Solution: $x^2 + 14x - 15 = 0$

Transposing 15 to the right member,

$$x^2 + 14x = 15$$

Half of 14 is 7. Adding the square of 7 to each member,

$$x^2 + 14x + 7^2 = 15 + 49$$

Expressing as a square,

$$(x + 7)^2 = 64$$

Taking the square root of each member,

$$x + 7 = \pm 8$$

$$x + 7 = + 8$$

$$x = 8 - 7$$

$$x = 1 \quad \text{Ans.}$$

$$x + 7 = - 8$$

$$x = - 8 - 7$$

$$x = - 15 \quad \text{Ans.}$$

Check: $x = 1$

Left

$$x^2 + 14x - 15$$

$$1 + 14 - 15$$

$$15 - 15$$

Right

$$0$$

$$0$$

$$0$$

Check: $x = - 15$

Left

$$x^2 + 14x - 15 =$$

$$225 - 210 - 15 =$$

$$225 - 225$$

Right

$$0$$

$$0$$

$$0$$

When the term in x^2 has a coefficient other than 1 each member of the equation must first be divided by that coefficient.

Example 2: Solve and check:

$$2x^2 + 5x - 3 = 0$$

Solution:

$$2x^2 + 5x - 3 = 0$$

Dividing by 2,

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

Transposing,

$$x^2 + \frac{5}{2}x = \frac{3}{2}$$

Half of $\frac{5}{2}$ is $\frac{1}{2} \cdot \frac{5}{2} = \frac{5}{4}$

$$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16}$$

$$\left(x + \frac{5}{4}\right)^2 = \frac{49}{16}$$

$$x + \frac{5}{4} = \pm \frac{7}{4}$$

$$x + \frac{5}{4} = \frac{7}{4}$$

$$x = -\frac{1}{2} \quad \text{Ans.}$$

$$x + \frac{5}{4} = -\frac{7}{4}$$

$$x = -3 \quad \text{Ans.}$$

Check: $x = \frac{1}{2}$

Check: $x = -3$

Left

Right

Left

Right

$$2x^2 + 5x - 3$$

$$0$$

$$2x^2 + 5x - 3$$

$$0$$

$$2\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) - 3$$

$$0$$

$$2(9) + 5(-3) - 3$$

$$0$$

$$\frac{1}{2} + \frac{5}{2} - 3$$

$$0$$

$$18 - 15 - 3$$

$$0$$

$$3 - 3$$

$$0$$

$$18 - 18$$

$$0$$

$$0 \longleftrightarrow 0$$

$$0 \longleftrightarrow 0$$

EXAMPLE 133

Solve by completing the square and check each answer:

$$1. x^2 + 6x + 9 = 100$$

$$6. r^2 - 2r - 4 = 0$$

$$2. x^2 + 6x = 16$$

$$\text{Ans. } r = 1 \pm \sqrt{5}$$

$$3. y^2 - 8y - 20 = 0$$

or

$$4. s^2 = 24s - 80$$

$$1 - \sqrt{5}$$

$$5. x^2 - 10x = 9975$$

$$7. x^2 + 3x = 4$$

8. $n^2 - 18 = 7n$

9. $104 - 5s = s^2$

10. $x^2 + \frac{2}{3}x = 11$

Hint: Half of $\frac{2}{3}$ is ?

11. $3x^2 + 4x = 7$

12. $2z^2 - 3z + 1 = 0$

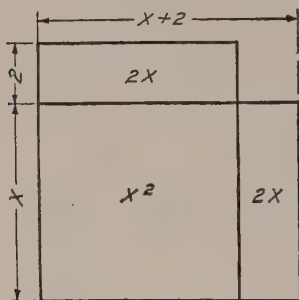
13. $t^2 - 8t = 0$

14. $y^2 + 12y = -36$

15. $4x^2 + 1 = 8x$

Ans. $x = 1 \pm \frac{1}{2}\sqrt{3}$

16. About 1500 years ago the best mathematicians in the world were the Hindus, from whom through the Arabs we



AREA OF DIAGRAM = 32

obtained the so-called Arabic numerals. These Hindus solved quadratic equations by means of a diagram, completing the square as follows:

To solve $x^2 + 4x = 32$

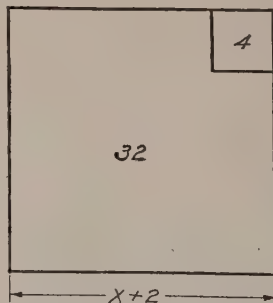
After drawing any square to represent x^2 , the term in x ($4x$) was divided into two equal parts, each giving a strip ($2x$) to be placed along a side of the square.

To complete the square the upper right corner must be filled in, thus adding 2×2 or 4.

It then becomes clear that $x + 2$ is the side of a square whose area is $32 + 4$ or 36, so that $x + 2$ must be 6.

If $x + 2 = 6$, then $x = 4$

Now solve the equation $x^2 + 4x = 32$ in the ordinary way, getting a negative answer also. (The latter was not known to the Hindus.)



AREA OF DIAGRAM = 32 + 4 OR 36

17. Draw diagrams similar to those in Example 16 and use them to solve $x^2 + 8x = 84$.

18. Solve for x : $x^2 + 2rx = 3r^2$. Check each answer.

19. Solve for x : $x^2 + 2px = q$. Ans. $x = -p \pm \sqrt{p^2 + q}$

20. Solve for x : $x^2 - ax = b$. Ans. $x = \frac{a \pm \sqrt{a^2 + 4b}}{2}$

21. Find a number such that its square is 3 less than 4 times the number.

22. I am thinking of a number such that when multiplied by a number 10 larger than itself, the product is 56. What is the number?

23. Solve for x : $ax^2 + bx + c = 0$
 Ans. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

SOLUTION BY FORMULA

The results of the last example of the preceding exercise may be taken as a formula, since every quadratic equation can be simplified and expressed as of the type form

$$ax^2 + bx + c = 0.$$

The method of completing the square is seldom used in everyday work, and its chief importance is that it enables us to understand the derivation of the formula.

Examples: Simplify and express the given equations in the standard form $ax^2 + bx + c = 0$.

$$(a) \quad 7 + 3x = x - x^2$$

$$\text{Transposing,} \quad x^2 + 3x - x + 7 = 0$$

$$\text{Collecting terms,} \quad x^2 + 2x - 7 = 0$$

$$[\text{Here } a = 1, b = 2, c = -7]$$

$$(b) \quad 2x(x - 3) = (x - 5)(x + 5)$$

$$2x^2 - 6x = x^2 - 25$$

$$2x^2 - x^2 - 6x + 25 = 0$$

$$x^2 - 6x + 25 = 0$$

$$[a = 1, b = -6, c = 25]$$

Oral Exercise

Do not try to solve these equations, but merely to simplify into standard form as above, and to give the values of a , b , and c :

1. $3x^2 - 5x + 2 = 0$

4. $5x = 8 - 2x^2$

2. $6x^2 + x - 2 = 0$

5. $x^2 + 7x = 3(x + 7)$

3. $7x^2 - x = 1$

6. $mx^2 = x + r$

In the following examples:

First. Express the equation in standard form.

Second. Find x by substituting the values of a , b , and c for that particular equation in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Third. Check each answer separately.

Example 1:

$$3x^2 + 7x + 2 = 0$$

Solution:

$$a = 3, b = 7, c = 2$$

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \cdot 3 \cdot 2}}{6} \\ &= \frac{-7 \pm \sqrt{49 - 24}}{6} \\ &= \frac{-7 \pm \sqrt{25}}{6} \\ &= \frac{-7 \pm 5}{6} \end{aligned}$$

First Answer.

$$x = \frac{-7 + 5}{6}$$

$$x = \frac{-2}{6}$$

$$x = -\frac{1}{3} \quad \text{Ans.}$$

Second Answer:

$$x = \frac{-7 - 5}{6}$$

$$x = \frac{-12}{6}$$

$$x = -2 \quad \text{Ans.}$$

Check: $x = -\frac{1}{3}$

Check: $x = -2$

Left	Right
$3x^2 + 7x + 2$	0
$3(\frac{1}{9}) + 7(-\frac{1}{3}) + 2$	0
$\frac{1}{3} - \frac{7}{3} + 2$	0
$0 \longleftrightarrow 0$	

Left	Right
$3x^2 + 7x + 2$	0
$3(4) + 7(-2) + 2$	0
$12 - 14 + 2$	0
$0 \longleftrightarrow 0$	

Example 2: $2x^2 - x - 6 = 0$

Solution:

$$a = 2, b = -1, c = -6$$

$$\begin{aligned}
 x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-12)}}{4} \\
 &= \frac{1 \pm \sqrt{1 + 48}}{4} \\
 &= \frac{1 \pm \sqrt{49}}{4} \\
 &= \frac{1 \pm 7}{4}
 \end{aligned}$$

First Answer:

$$x = \frac{1 + 7}{4}$$

$$x = \frac{8}{4}$$

$$x = 2 \quad \text{Ans.}$$

Second Answer:

$$x = \frac{1 - 7}{4}$$

$$x = \frac{-6}{4}$$

$$x = -\frac{3}{2} \quad \text{Ans.}$$

Check: $x = 2$	
Left	Right
$2x^2 - x - 6$	0
$2(4) - 2 - 6$	0
$8 - 2 - 6$	0
$0 \longleftrightarrow 0$	

Check: $x = -\frac{3}{2}$	
Left	Right
$2x^2 - x - 6$	0
$2(\frac{9}{4}) - (-\frac{3}{2}) - 6$	0
$\frac{9}{2} + \frac{3}{2} - 6$	0
$0 \longleftrightarrow 0$	

Example 3: $8 - 6h = 1 - h^2$

Solution: Transposing, $h^2 - 6h + 8 - 1 = 0$

$$h^2 - 6h + 7 = 0$$

In using the formula, h takes the place of x .

$$a = 1, b = -6, c = 7$$

$$\begin{aligned} h &= \frac{-(-6) \pm \sqrt{36 - 4(7)}}{2} \\ &= \frac{6 \pm \sqrt{36 - 28}}{2} \\ &= \frac{6 \pm \sqrt{8}}{2} \\ &= \frac{6 \pm 2\sqrt{2}}{2} \\ &= 3 \pm \sqrt{2} \end{aligned}$$

First Answer:

$$h = 3 + \sqrt{2}$$

Second Answer:

$$h = 3 - \sqrt{2}$$

Check: $h = 3 + \sqrt{2}$

Left

Right

$$8 - 6h$$

$$1 - h^2$$

$$8 - 6(3 + \sqrt{2})$$

$$1 - (3 + \sqrt{2})^2$$

$$8 - 18 - 6\sqrt{2}$$

$$1 - (9 + 6\sqrt{2} + 2)$$

$$8 - 18 - 6\sqrt{2}$$

$$1 - 9 - 6\sqrt{2} - 2$$

$$-10 - 6\sqrt{2} \longleftrightarrow -10 - 6\sqrt{2}$$

The check for $x = 3 - \sqrt{2}$ is left for the pupil to perform.

Caution: Before substituting in the formula, all terms must be transposed to the left so that the right member is zero and the equation is in standard form.

EXERCISE 134

Solve by the formula, and check:

1. $x^2 + 6x + 8 = 0$

8. $12x^2 - 5x = 2$

2. $3x^2 + 7x + 4 = 0$

9. $n + 2 = 6n^2$

3. $4x^2 + 5x + 1 = 0$

10. $20 = t(9 - t)$

4. $x^2 - 5x + 6 = 0$

11. $x(2x - 1) = 15$

5. $3n^2 - 10n + 7 = 0$

12. $10y^2 + 3 = 11y$

6. $5y^2 + y - 4 = 0$

13. $x : 5 = 2 : 3 + x$

7. $7h^2 = 6 - h$

14. $w(w + 4) = 4(2w - 1)$

15. $7 : s + 5 = s + 5 : 2s + 3$

16. $R^2 + (R + 3)^2 = 225$

19. $\frac{1}{3}x - 1 = \frac{x}{6}(1 - x)$

17. $\frac{z + 3}{z + 1} = \frac{3z + 7}{z + 4}$

20. $1 + \frac{20}{l} = \frac{10}{l - 3}$

18. $\frac{x}{2} - \frac{1}{3} = \frac{2x + 4}{x}$

21. $\frac{1}{n} - \frac{1}{n + 1} = \frac{1}{12}$

22. $2x^2 + 9x + 3 = 0$. Answer in radical form. Omit check.

23. $x^2 + 4x - 1 = 0$. Ans. $x = -2 \pm \sqrt{5}$.

24. $r^2 = 6r + 2$. Simplify the radical answer, and check.

25. $8 - 8x = x^2$. Simplify and check.

26. $px^2 + qx + r = 0$. Omit check.

27. $3x^2 - 8x = 0$. Hint: What is the value of c ?

28. (a) Solve by completing the square $x^2 + 4x = 96$.

(b) Check by solving the same equation by the formula.

29. Solve by formula and also by completing the square:

$3n^2 = n + 2$.

30. Solve for x by the formula and by completing the square: $x^2 - 2mx = q$.

31. Solve for x by either method: $6x^2 - 6a^2 = 5ax$.

32. Solve for x by formula: $4q^2 = 12qx - 9x^2$.

33. Solve for x by formula: $px^2 + x(1 + pq) + q = 0$

Hint: $a = p$, $b = 1 + pq$, etc.

In Examples 34 to 41 obtain the value of each answer to three significant figures, getting the square roots from the tables. Check one answer in each example.

$$34. x^2 + 5x + 3 = 0$$

Solution:

$$x = \frac{-5 \pm \sqrt{25 - 12}}{2}$$

$$x = \frac{-5 \pm \sqrt{13}}{2}$$

First Answer:

$$x = \frac{-5 + 3.606}{2}$$

$$x = \frac{-1.394}{2}$$

$$x = -.697$$

Second Answer:

$$x = \frac{-5 - 3.606}{2}$$

$$x = \frac{-8.606}{2}$$

$$x = -4.303$$

$$x = -4.30 \text{ (to 3 figures)}$$

Check: $x = -0.697$

Left

Right

$$\begin{array}{r|l} x^2 + 5x + 3 & 0 \\ (-.697)^2 + 5(-.697) + 3 & 0 \\ .486 - 3.485 + 3 & 0 \\ 3.486 - 3.485 & \leftrightarrow 0 \end{array}$$

Check: $x = -4.30$

Left

Right

$$\begin{array}{r|l} x^2 + 5x + 3 & 0 \\ (-4.30)^2 + 5(-4.30) + 3 & 0 \\ 18.49 - 21.50 + 3 & 0 \\ 21.49 - 21.50 & \leftrightarrow 0 \end{array}$$

Approximately

Approximately

Note that each answer checks to three figures.

$$35. x^2 + 7x + 2 = 0$$

$$38. 2x^2 + x = 0.2$$

$$36. x^2 + x - 1 = 0$$

$$39. x^2 = 8x - 3$$

$$37. r^2 - 2r - 3.14 = 0$$

$$40. 17 - x^2 = 7 - x$$

$$41. x^2 + 0.7x = 5$$

42. Three times the square of a certain number is 11 more than half the number. Find the number.

43. Find two numbers such that one exceeds the other by 8, and the sum of their squares is 130.

44. Find two consecutive numbers whose product is 156.

SOLUTION BY FACTORING

Oral Exercise

1. Could you tell at sight the value of the quantity $abcd$ if $a = 89$, $b = 57.3$, $c = 0$, and $d = 5$?

2. Here are two sets of values which satisfy the equation $xy = 12$. Write three additional sets.

	x	y
(1)	-6	-2
(2)	$\frac{1}{2}$	24
(3)	_____	_____
(4)	_____	_____
(5)	_____	_____

3. Given $xy = 0$. Write four sets of values for x and y .

In the second example you cannot tell definitely what either of the numbers will be unless you know the value of the other one.

In the third example you know in advance that if the product is to be zero, one or the other of the numbers must itself be zero. (Zero is perhaps the most interesting of all numbers, and the most valuable in mathematical work.)

4. If $yz = 0$, what must be the value of either y or z ?

5. If $5x = 0$, what is the value of x ?

6. If $x(x - 4) = 0$, what must be the value of either x or $x - 4$?

7. What would be the value of $(x + 3)(x - 2)$ if either $x + 3 = 0$ or $x - 2 = 0$?

8. What values of x would make $(x + 3)(x - 2) = 0$?

Principle. The product of several factors will be zero if any factor is zero.

Example: Solve by factoring $x^2 + 20 = 9x$.

Solution: Transpose so that the right member will be zero:

$$x^2 - 9x + 20 = 0$$

Factor

$$(x - 5)(x - 4) = 0$$

This product $(x - 5)(x - 4)$ will be zero if either of its factors is zero.

Therefore the equation will be true if

$$\begin{aligned}\text{either } x - 5 &= 0 \\ x &= 5\end{aligned}$$

$$\begin{aligned}\text{or } x - 4 &= 0 \\ x &= 4\end{aligned}$$

Instead of checking by substituting the answers, one may check by solving the original example by a different method, either by the formula or by completing the square.

It should be noticed that when factoring, just as when using the formula, the equation must first be arranged in standard form.

EXERCISE 135

Solve by the method of factoring, and check:

1. $x^2 - 7x + 10 = 0$
2. $x^2 + 8x + 12 = 0$
3. $y^2 - y - 6 = 0$
4. $x^2 - 16 = 0$
5. $r^2 + 5r + 4 = 0$
6. $s^2 + 9 = 6s$
7. $x^2 - 10x = -21$
8. $3x^2 - 5x + 2 = 0$
9. $7n^2 - 6 = 11n$
10. $5 - 6x = -x^2$
11. $57 = 16t + t^2$
12. $3x^2 = 15x - 18$
13. $M^2 + 6 = 7M$
14. $14 + 12x = 2x^2$
15. $x^2 - 3x = 0$
16. $10y^2 = 3y$
17. $x^2 + x = x + 1$
18. $0 = 6w^2 - 5w + 1$
19. $4l^2 - 12l = -5$
20. $3z^2 + 13z = 10$
21. Solve by factoring, $x^2 - 12 = 4x$
22. Solve by formula, $x^2 - 12 = 4x$
23. Solve by completing the square, $x^2 - 12 = 4x$
24. Solve for x by completing the square, $x^2 - 2ax = b^2 - a^2$

Solve the remaining examples by factoring or by either of the other two methods, whichever you prefer. Check.

25. $100x^2 - 20x + 1 = 0$

26. $x^2 = \frac{2}{3}x$

$$27. y = \frac{3}{4} y^2$$

$$28. (x + 1)^2 = (2x - 1)(x - 1)$$

$$29. n^2 + (n + 4)^2 = 58$$

$$30. x : 3x - 1 = 3 : x + 5$$

$$31. (t + 2)(t + 5) = 40$$

$$32. (x + 3)(x - 3) - (2x - 1)(x - 2) = 21$$

$$33. v : 2 = 16 : 12 - v$$

$$34. \frac{1}{x} - \frac{1}{x + 1} = \frac{1}{2}$$

$$35. \frac{m + 3}{m} - \frac{m}{m + 3} = \frac{15}{4}$$

$$36. \frac{5}{2y} + \frac{8}{2y + 1} = 9$$

$$37. \text{Solve for } x, x^2 + 5ax + 4a^2 = 0$$

$$38. \text{Solve for } y, ay + 3a^2 = 2y^2$$

$$39. \text{Solve for } x, x^2 - 9b^2 = 0$$

$$40. \text{Solve for } r, r^2 + pr = 0$$

$$41. n + \frac{1}{n - 3} = 5$$

$$42. \frac{w}{w + 1} = 1 - \frac{w}{w + 4}$$

$$43. 2x - 1 : x + 1 = x + 1 : x - 2$$

$$\text{Omit check. } x = \frac{7 \pm 3\sqrt{5}}{2} \quad \text{Ans.}$$

$$44. s + 2 - \frac{6}{s + 2} = 1 \qquad 46. \frac{x + 8}{x - 8} - 2 = \frac{24}{x - 4}$$

$$45. \frac{8}{y + 6} + \frac{12 - y}{y - 6} = 1 \qquad 47. \frac{x + 5}{x - 5} + \frac{x - 5}{x + 5} = \frac{10}{3}$$

Problems Solved by Quadratic Equations.

No special directions are needed either for representing the various quantities in these problems or for translating the

verbal statements into equations. In regard to the results, however, the equation may give two answers when only one answer fits the problem. The superfluous answer is easily rejected by common sense and by checking answers in the statement of the problem rather than in your equation.

EXERCISE 136

1. In Classroom 217 the number of seats in a row is 4 more than the number of rows, the total number of seats being 77. How many rows are there?

2. What number is equal to its reciprocal? (Only one answer.)

3. What number exceeds its reciprocal by $1\frac{1}{2}$? (Two answers.)

4. Find two numbers whose difference is 10 and whose product is 56. (Two sets of answers.)

5. The square of a number is 4 more than 3 times the number. What is the number?

6. By what number can 96 be divided so that the quotient exceeds the divisor by 4?

7. Find a number which exceeds its square by $\frac{1}{4}$.

8. The area of a rectangle is 48 square feet and its perimeter is 38 feet. Find the length and width.

9. The lengths of the three sides of a right triangle (in inches) are consecutive numbers. What are they?

10. The diagonal of a rectangle exceeds twice the width by 3". Find the width if the length of the rectangle is 12".

11. The diagonal of a square is 4" longer than the side. Find the length of the side. (To three figures.)

12. A ranch owner said, "I own a square ranch which contains just as many square miles of land as there are miles of fence surrounding it." Can you tell how large the ranch is?

13. A dealer sells a bicycle for \$39, gaining as many per

cent as the bicycle cost him in dollars. How many dollars did it cost him?

14. By taking space for a walk w feet wide along one end and a side of a lawn $50' \times 40'$ the area is reduced to 1575 square feet. Find w .

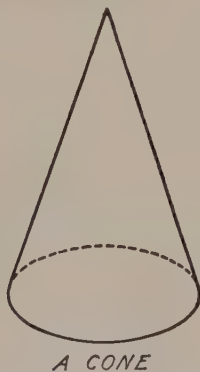
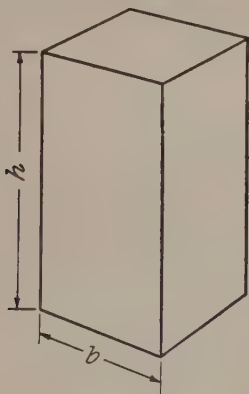
15. A picture $24'' \times 15''$ has around it a frame w inches wide. Find w if the surface of the frame is $\frac{3}{4}$ that of the picture itself.

16. By the addition of f feet to both its length and width, a rectangle $30' \times 40'$ was doubled in area. Find f (to three figures).

17. The distance passed through by an object thrown downwards with a velocity of v feet per second is given by the formula $d = vt + 16t^2$. Find t , if the distance is $56'$ and $v = 40'$.

18. From the preceding formula find t to two significant figures, if $v = 10'$ and the distance is $100'$. Ans. $t = 2.2$

19. The formula for the surface of a square prism is $S = 2b^2 + 4bh$. Find b if $S = 210$ and $h = 8$



20. The surface of a cone is given by the formula $S = \pi R^2 + \pi RL$. Find the radius if $S = 374$, $l = 10$, and $\pi = \frac{22}{7}$.

21. The surface of a cylinder is given by the formula $S = 2\pi R^2 + 2\pi Rh$. Find R to two significant figures if $S = 200$, $h = 8$, and $\pi = \frac{22}{7}$.

22. The climb up Camel's Hump in the Green Mountains is 6 miles long. A scout averaged twice as fast a rate coming down as he did going up, and his time for the round trip was six hours. What was his rate up the mountain?

23. From New York to Chicago, a distance of 900 miles, an army plane traveled 100 miles an hour faster than the fastest train. What is the rate of the train?

24. Between Boston and New York, 240 miles apart, the "Limited" train averages 8 miles per hour faster than the regular trains, and takes one hour less time. Find the rate of the regular trains.

APPENDIX



TABLES

TABLE 1. SQUARES, CUBES, SQUARE ROOTS AND CUBE ROOTS
OF NUMBERS FROM 1 to 50.

NUMBER n	SQUARE n^2	CUBE n^3	SQUARE ROOT $\sqrt{n}$	CUBE ROOT $\sqrt[3]{n}$
1	1	1	1.000	1.000
2	4	8	1.414	1.260
3	9	27	1.732	1.442
4	16	64	2.000	1.587
5	25	125	2.236	1.710
6	36	216	2.449	1.817
7	49	343	2.646	1.913
8	64	512	2.828	2.000
9	81	729	3.000	2.080
10	100	1,000	3.162	2.154
11	121	1,331	3.317	2.224
12	144	1,728	3.464	2.289
13	169	2,197	3.606	2.351
14	196	2,744	3.742	2.410
15	225	3,375	3.873	2.466
16	256	4,096	4.000	2.520
17	289	4,913	4.123	2.571
18	324	5,832	4.243	2.621
19	361	6,859	4.359	2.668
20	400	8,000	4.472	2.714
21	441	9,261	4.583	2.759
22	484	10,648	4.690	2.802
23	529	12,167	4.796	2.844
24	576	13,824	4.899	2.884
25	625	15,625	5.000	2.924
26	676	17,576	5.099	2.962
27	729	19,683	5.196	3.000
28	784	21,952	5.291	3.037
29	841	24,389	5.385	3.072
30	900	27,000	5.477	3.107
31	961	29,791	5.568	3.141
32	1,024	32,768	5.657	3.175
33	1,089	35,937	5.745	3.208
34	1,156	39,304	5.831	3.240
35	1,225	42,875	5.916	3.271
36	1,296	46,656	6.000	3.302
37	1,369	50,653	6.083	3.332
38	1,444	54,872	6.164	3.362
39	1,521	59,319	6.245	3.391
40	1,600	64,000	6.325	3.420
41	1,681	68,921	6.403	3.448
42	1,764	74,088	6.481	3.476
43	1,849	79,507	6.557	3.503
44	1,936	85,184	6.633	3.530
45	2,025	91,125	6.708	3.557
46	2,116	97,336	6.782	3.583
47	2,209	103,823	6.856	3.609
48	2,304	110,592	6.928	3.634
49	2,401	117,649	7.000	3.659
50	2,500	125,000	7.071	3.684

TABLE 2. SQUARE ROOTS OF NUMBERS FROM 1.00 TO 9.99

(Use this table also for numbers from $\widehat{100}$ to $\widehat{999}$ and from $\widehat{.0100}$ to $\widehat{.0999}$, etc., placing the decimal point carefully.)

N	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022
4.1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343

TABLE 2. SQUARE ROOTS OF NUMBERS FROM 1.00 TO 9.99 (continued)

N	0	1	2	3	4	5	6	7	8	9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.878	2.879
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048
9.3	3.050	3.051	3.053	3.054	3.056	3.058	3.059	3.061	3.063	3.064
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161

TABLE 3. SQUARE ROOTS OF NUMBERS FROM 10.0 to 99.9

(Use this table also for numbers from $\widehat{1000}$ to $\widehat{9999}$, and from $\widehat{.1000}$ to $\widehat{.9999}$, etc., taking care to place the decimal point.)

N	0	1	2	3	4	5	6	7	8	9
10.	3.162	3.178	3.194	3.210	3.225	3.240	3.256	3.271	3.286	3.302
11.	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450
12.	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592
13.	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728
14.	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860
15.	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987
16.	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111
17.	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231
18.	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347
19.	4.359	4.370	4.382	4.393	4.404	4.416	4.427	4.438	4.450	4.461
20.	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572
21.	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680
22.	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785
23.	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889
24.	4.899	4.909	4.919	4.929	4.940	4.950	4.960	4.970	4.980	4.990
25.	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089
26.	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187
27.	5.196	5.206	5.215	5.225	5.234	5.244	5.254	5.263	5.273	5.282
28.	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376
29.	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468
30.	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559
31.	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648
32.	5.657	5.666	5.674	5.683	5.692	5.701	5.710	5.718	5.727	5.736
33.	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822
34.	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.907
35.	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992
36.	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075
37.	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156
38.	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237
39.	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317
40.	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395
41.	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473
42.	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550
43.	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626
44.	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701
45.	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775
46.	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848
47.	6.856	6.863	6.870	6.878	6.885	6.892	6.899	6.907	6.914	6.921
48.	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993
49.	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064
50.	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134
51.	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204
52.	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273
53.	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342
54.	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409

TABLE 3. SQUARE ROOTS OF NUMBERS FROM 10.0 to 99.9 (Continued)

N	0	1	2	3	4	5	6	7	8	9
55.	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477
56.	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543
57.	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609
58.	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675
59.	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740
60.	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804
61.	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868
62.	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931
63.	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994
64.	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056
65.	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118
66.	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179
67.	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240
68.	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301
69.	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361
70.	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420
71.	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479
72.	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538
73.	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597
74.	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654
75.	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712
76.	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769
77.	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826
78.	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883
79.	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939
80.	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994
81.	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050
82.	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105
83.	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160
84.	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214
85.	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268
86.	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322
87.	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375
88.	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429
89.	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482
90.	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534
91.	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586
92.	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638
93.	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690
94.	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742
95.	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793
96.	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844
97.	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894
98.	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945
99.	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995

TABLE 4. NATURAL TANGENTS

AN- GLE	0'	10'	20'	30'	40'	50'	60'	AN- GLE
0°	.0000	.0029	.0058	.0087	.0116	.0145	.0175	89°
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88
2	.0349	.0378	.0407	.0437	.0466	.0495	.0524	87
3	.0524	.0553	.0582	.0612	.0641	.0670	.0699	86
4	.0699	.0729	.0758	.0787	.0816	.0846	.0875	85
5	.0875	.0904	.0934	.0963	.0992	.1022	.1051	84
6	.1051	.1080	.1110	.1139	.1169	.1198	.1228	83
7	.1228	.1257	.1287	.1317	.1346	.1376	.1405	82
8	.1405	.1435	.1465	.1495	.1524	.1554	.1584	81
9	.1584	.1614	.1644	.1673	.1703	.1733	.1763	80
10	.1763	.1793	.1823	.1853	.1883	.1914	.1944	79
11	.1944	.1974	.2004	.2035	.2065	.2095	.2126	78
12	.2126	.2156	.2186	.2217	.2247	.2278	.2309	77
13	.2309	.2339	.2370	.2401	.2432	.2462	.2493	76
14	.2493	.2524	.2555	.2586	.2617	.2648	.2679	75
15	.2679	.2711	.2742	.2773	.2805	.2836	.2867	74
16	.2867	.2899	.2931	.2962	.2994	.3026	.3057	73
17	.3057	.3089	.3121	.3153	.3185	.3217	.3249	72
18	.3249	.3281	.3314	.3346	.3378	.3411	.3443	71
19	.3443	.3476	.3508	.3541	.3574	.3607	.3640	70
20	.3640	.3673	.3706	.3739	.3772	.3805	.3839	69
21	.3839	.3872	.3906	.3939	.3973	.4006	.4040	68
22	.4040	.4074	.4108	.4142	.4176	.4210	.4245	67
23	.4245	.4279	.4314	.4348	.4383	.4417	.4452	66
24	.4452	.4487	.4522	.4557	.4592	.4628	.4663	65
25	.4663	.4699	.4734	.4770	.4806	.4841	.4877	64
26	.4877	.4913	.4950	.4986	.5022	.5059	.5095	63
27	.5095	.5132	.5169	.5206	.5243	.5280	.5317	62
28	.5317	.5354	.5392	.5430	.5467	.5505	.5543	61
29	.5543	.5581	.5619	.5658	.5696	.5735	.5774	60
30	.5774	.5812	.5851	.5890	.5930	.5969	.6009	59
31	.6009	.6048	.6088	.6128	.6168	.6208	.6249	58
32	.6249	.6289	.6330	.6371	.6412	.6453	.6494	57
33	.6494	.6536	.6577	.6619	.6661	.6703	.6745	56
34	.6745	.6787	.6830	.6873	.6916	.6959	.7002	55
35	.7002	.7046	.7089	.7133	.7177	.7221	.7265	54
36	.7265	.7310	.7355	.7400	.7445	.7490	.7536	53
37	.7536	.7581	.7627	.7673	.7720	.7766	.7813	52
38	.7813	.7860	.7907	.7954	.8002	.8050	.8098	51
39	.8098	.8146	.8195	.8243	.8292	.8342	.8391	50
40	.8391	.8441	.8491	.8541	.8591	.8642	.8693	49
41	.8693	.8744	.8796	.8847	.8899	.8952	.9004	48
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46
44	.9657	.9713	.9770	.9827	.9884	.9942	1.0000	45
	60'	50'	40'	30'	20'	10'	0'	AN- GLE

NATURAL COTANGENTS.

TABLE 4. NATURAL TANGENTS (Continued)

AN- GLE	0'	10'	20'	30'	40'	50'	60'	AN- GLE
45°	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44°
46	1.0355	1.0416	1.0477	1.0538	1.0599	1.0661	1.0724	43
47	1.0724	1.0786	1.0850	1.0913	1.0977	1.1041	1.1106	42
48	1.1106	1.1171	1.1237	1.1303	1.1369	1.1436	1.1504	41
49	1.1504	1.1571	1.1640	1.1708	1.1778	1.1847	1.1918	40
50	1.1918	1.1988	1.2059	1.2131	1.2203	1.2276	1.2349	39
51	1.2349	1.2423	1.2497	1.2572	1.2647	1.2723	1.2799	38
52	1.2799	1.2876	1.2954	1.3032	1.3111	1.3190	1.3270	37
53	1.3270	1.3351	1.3432	1.3514	1.3597	1.3680	1.3764	36
54	1.3764	1.3848	1.3934	1.4019	1.4106	1.4193	1.4281	35
55	1.4281	1.4370	1.4460	1.4550	1.4641	1.4733	1.4826	34
56	1.4826	1.4919	1.5013	1.5108	1.5204	1.5301	1.5399	33
57	1.5399	1.5497	1.5597	1.5697	1.5798	1.5900	1.6003	32
58	1.6003	1.6107	1.6212	1.6319	1.6426	1.6534	1.6643	31
59	1.6643	1.6753	1.6864	1.6977	1.7090	1.7205	1.7321	30
60	1.7321	1.7437	1.7556	1.7675	1.7796	1.7917	1.8040	29
61	1.8040	1.8165	1.8291	1.8418	1.8546	1.8676	1.8807	28
62	1.8807	1.8940	1.9074	1.9210	1.9347	1.9486	1.9626	27
63	1.9626	1.9768	1.9912	2.0057	2.0204	2.0353	2.0503	26
64	2.0503	2.0655	2.0809	2.0965	2.1123	2.1283	2.1445	25
65	2.1445	2.1609	2.1775	2.1943	2.2113	2.2286	2.2460	24
66	2.2460	2.2637	2.2817	2.2998	2.3183	2.3369	2.3559	23
67	2.3559	2.3750	2.3945	2.4142	2.4342	2.4545	2.4751	22
68	2.4751	2.4960	2.5172	2.5386	2.5605	2.5826	2.6051	21
69	2.6051	2.6279	2.6511	2.6746	2.6985	2.7228	2.7475	20
70	2.7475	2.7725	2.7980	2.8239	2.8502	2.8770	2.9042	19
71	2.9042	2.9319	2.9600	2.9887	3.0178	3.0475	3.0777	18
72	3.0777	3.1084	3.1397	3.1716	3.2041	3.2371	3.2703	17
73	3.2709	3.3052	3.3402	3.3759	3.4124	3.4495	3.4874	16
74	3.4874	3.5261	3.5656	3.6059	3.6470	3.6891	3.7321	15
75	3.7321	3.7760	3.8208	3.8667	3.9136	3.9617	4.0108	14
76	4.0108	4.0611	4.1126	4.1653	4.2193	4.2747	4.3315	13
77	4.3315	4.3897	4.4494	4.5107	4.5736	4.6382	4.7046	12
78	4.7046	4.7729	4.8430	4.9152	4.9894	5.0658	5.1446	11
79	5.1446	5.2257	5.3093	5.3955	5.4845	5.5764	5.6713	10
80	5.6713	5.7694	5.8708	5.9758	6.0844	6.1970	6.3138	9
81	6.3138	6.4348	6.5606	6.6912	6.8269	6.9682	7.1154	8
82	7.1154	7.2687	7.4287	7.5958	7.7704	7.9530	8.1443	7
83	8.1443	8.3450	8.5555	8.7769	9.0098	9.2553	9.5144	6
84	9.5144	9.7882	10.0780	10.3854	10.7119	11.0594	11.4301	5
85	11.4301	11.8262	12.2505	12.7062	13.1969	13.7267	14.3007	4
86	14.3007	14.9244	15.6048	16.3499	17.1693	18.0750	19.0311	3
87	19.0811	20.2056	21.4704	22.9038	24.5418	26.4316	28.6363	2
88	28.6363	31.2416	34.3678	38.1885	42.9641	49.1039	57.2900	1
89	57.2900	68.7501	85.9398	114.5887	171.8854	343.7737	Infinite	0
	60'	50'	40'	30'	20'	10'	0'	AN- GLE.

NATURAL COTANGENTS

TABLE 5. NATURAL SINES

AN- GLE	0'	10'	20'	30'	40'	50'	60'	AN- GLE
0°	.0000	.0029	.0058	.0087	.0116	.0145	.0175	89°
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88
2	.0349	.0378	.0407	.0436	.0465	.0494	.0523	87
3	.0523	.0552	.0581	.0610	.0640	.0669	.0698	86
4	.0698	.0727	.0756	.0785	.0814	.0843	.0872	85
5	.0872	.0901	.0929	.0958	.0987	.1016	.1045	84
6	.1045	.1074	.1103	.1132	.1161	.1190	.1219	83
7	.1219	.1248	.1276	.1305	.1334	.1363	.1392	82
8	.1392	.1421	.1449	.1478	.1507	.1536	.1564	81
9	.1564	.1593	.1622	.1650	.1679	.1708	.1736	80
10	.1736	.1765	.1794	.1822	.1851	.1880	.1908	79
11	.1908	.1937	.1965	.1994	.2022	.2051	.2079	78
12	.2079	.2108	.2136	.2164	.2193	.2221	.2250	77
13	.2250	.2278	.2306	.2334	.2362	.2391	.2419	76
14	.2419	.2447	.2476	.2504	.2532	.2560	.2588	75
15	.2588	.2616	.2644	.2672	.2700	.2728	.2756	74
16	.2756	.2784	.2812	.2840	.2868	.2896	.2924	73
17	.2924	.2952	.2979	.3007	.3035	.3062	.3090	72
18	.3090	.3118	.3145	.3173	.3201	.3228	.3256	71
19	.3256	.3283	.3311	.3338	.3365	.3393	.3420	70
20	.3420	.3448	.3475	.3502	.3529	.3557	.3584	69
21	.3584	.3611	.3638	.3665	.3692	.3719	.3746	68
22	.3746	.3773	.3800	.3827	.3854	.3881	.3907	67
23	.3907	.3934	.3961	.3987	.4014	.4041	.4067	66
24	.4067	.4094	.4120	.4147	.4173	.4200	.4226	65
25	.4226	.4253	.4279	.4305	.4331	.4358	.4384	64
26	.4384	.4410	.4436	.4462	.4488	.4514	.4540	63
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60
30	.5000	.5025	.5050	.5075	.5100	.5125	.5150	59
31	.5150	.5175	.5200	.5225	.5250	.5275	.5299	58
32	.5299	.5324	.5348	.5373	.5398	.5422	.5446	57
33	.5446	.5471	.5495	.5519	.5544	.5568	.5592	56
34	.5592	.5616	.5640	.5664	.5688	.5712	.5736	55
35	.5736	.5760	.5783	.5807	.5831	.5854	.5878	54
36	.5878	.5901	.5925	.5948	.5972	.5995	.6018	53
37	.6018	.6041	.6065	.6088	.6111	.6134	.6157	52
38	.6157	.6180	.6202	.6225	.6248	.6271	.6293	51
39	.6293	.6316	.6338	.6361	.6383	.6406	.6428	50
40	.6428	.6450	.6472	.6494	.6517	.6539	.6561	49
41	.6561	.6583	.6604	.6626	.6648	.6670	.6691	48
42	.6691	.6713	.6734	.6756	.6777	.6799	.6820	47
43	.6820	.6841	.6862	.6884	.6905	.6926	.6947	46
44	.6947	.6967	.6988	.7009	.7030	.7050	.7071	45
	60'	50'	40'	30'	20'	10'	0'	AN- GLE

NATURAL COSINES.

TABLE 5. NATURAL SINES (Continued)

AN- GLE	0'	10'	20'	30'	40'	50'	60'	AN- GLE
45°	.7071	.7092	.7112	.7133	.7153	.7173	.7193	44°
46	.7193	.7214	.7234	.7254	.7274	.7294	.7314	43
47	.7314	.7333	.7353	.7373	.7392	.7412	.7431	42
48	.7431	.7451	.7470	.7490	.7509	.7528	.7547	41
49	.7547	.7566	.7585	.7604	.7623	.7642	.7660	40
50	.7660	.7679	.7698	.7716	.7735	.7753	.7771	39
51	.7771	.7790	.7808	.7826	.7844	.7862	.7880	38
52	.7880	.7898	.7916	.7934	.7951	.7969	.7986	37
53	.7986	.8004	.8021	.8039	.8056	.8073	.8090	36
54	.8090	.8107	.8124	.8141	.8158	.8175	.8192	35
55	.8192	.8208	.8225	.8241	.8258	.8274	.8290	34
56	.8290	.8307	.8323	.8339	.8355	.8371	.8387	33
57	.8387	.8403	.8418	.8434	.8450	.8465	.8480	32
58	.8480	.8496	.8511	.8526	.8542	.8557	.8572	31
59	.8572	.8587	.8601	.8616	.8631	.8646	.8660	30
60	.8660	.8675	.8689	.8704	.8718	.8732	.8746	29
61	.8746	.8760	.8774	.8788	.8802	.8816	.8829	28
62	.8829	.8843	.8857	.8870	.8884	.8897	.8910	27
63	.8910	.8923	.8936	.8949	.8962	.8975	.8988	26
64	.8988	.9001	.9013	.9026	.9038	.9051	.9063	25
65	.9063	.9075	.9088	.9100	.9112	.9124	.9135	24
66	.9135	.9147	.9159	.9171	.9182	.9194	.9205	23
67	.9205	.9216	.9228	.9239	.9250	.9261	.9272	22
68	.9272	.9283	.9293	.9304	.9315	.9325	.9336	21
69	.9336	.9346	.9356	.9367	.9377	.9387	.9397	20
70	.9397	.9407	.9417	.9426	.9436	.9446	.9455	19
71	.9455	.9465	.9474	.9483	.9492	.9502	.9511	18
72	.9511	.9520	.9528	.9537	.9546	.9555	.9563	17
73	.9563	.9572	.9580	.9588	.9596	.9605	.9613	16
74	.9613	.9621	.9628	.9636	.9644	.9652	.9659	15
75	.9659	.9667	.9674	.9681	.9689	.9696	.9703	14
76	.9703	.9710	.9717	.9724	.9730	.9737	.9744	13
77	.9744	.9750	.9757	.9763	.9769	.9775	.9781	12
78	.9781	.9787	.9793	.9799	.9805	.9811	.9816	11
79	.9816	.9822	.9827	.9833	.9838	.9843	.9848	10
80	.9848	.9853	.9858	.9863	.9868	.9872	.9877	9
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5
85	.9962	.9964	.9967	.9969	.9971	.9974	.9976	4
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1
89	.9998	.9999	.9999	1.0000	1.0000	1.0000	1.0000	0
	60'	50'	40'	30'	20'	10'	0'	AN- GLE

NATURAL COSINES.

TABLE 6. IMPORTANT UNITS OF MEASURE

Linear Measure

12 inches (12'')	= 1 foot (1')
3 feet	= 1 yard
5½ yards	= 1 rod
320 rods	= 1 mile
5280 feet	= 1 mile

Square Measure

144 sq. inches	= 1 sq. foot
9 sq. feet	= 1 sq. yard
30¼ sq. yards	= 1 sq. rod
160 sq. rods	= 1 acre
640 acres	= 1 sq. mile

Cubic Measure

1728 cu. inches	= 1 cu. foot
27 cu. feet	= 1 cu. yard

Liquid Measure

2 pints	= 1 quart
4 quarts	= 1 gallon
	(231 cubic inches)

Avoirdupois Weight

16 ounces	= 1 pound
2000 pounds	= 1 ton
2240 pounds	= 1 long ton

Dry Measure

8 quarts	= 1 peck
4 pecks	= 1 bushel
	(2150.4 cubic inches)

A few units of the metric system

Length:

10 millimeters	= 1 centimeter
10 centimeters	= 1 decimeter
10 decimeters	= 1 meter
1000 meters	= 1 kilometer

Some equivalents:

1 meter	= 39.37 inches
1 kilometer	= about $\frac{5}{8}$ mile
1 kilogram	= about $2\frac{1}{8}$ pounds
1 liter	= about 1.06 liquid quarts

Capacity:

1 cubic decimeter	} = 1 liter
or 1000 cubic centimeters	

Weight:

1 cu. centimeter of water	weighs 1 gram
1 liter of water	weighs 1 kilogram

The value of the number π , correct to seven significant figures, is 3.141593.

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